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From the Attic

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Abstract

We gather the following miscellaneous results in proof theory from the attic.

1. A provably well founded elementary ordering admits an elementary order preserving map.
2. A simple proof of an elementary bound for cut elimination in propositional calculus
3. Equivalents for Bar Induction, e.g., reflection schema for ω logic
4. Direct computations in an equatinal calculus PRE
5. Intuitionistic fixed point theories are conservative extensions of HA.
6. Proof theoretic strengths of classical fixed points theories
7. An equivalence between transfinite induction rule and iterated reflection schema over $I\Sigma_n$
8. Derivation lengths of finite rewrite rules reducing under lexicographic path orders and provably total functions in theories between $I\Sigma_1$ and $I\Sigma_2$

Each section can be read separately in principle.

1 Provably Well Founded Relations

In this section we show that if an elementary recursive relation $\prec$ is provably well founded in Peano Arithmetic PA, then there exists an *elementary recursive* order preserving map f of $\prec$ into an initial segment of ε_0. This gives an improvement on a result by Harrington and Takeuti (cf. [24], Theorem 13.6 and [10], p.33).

We say that a binary relation $\prec$ is *provably well founded* in PA if

$$PA(X) \vdash \forall n(\forall k \prec n X(k) \supset X(n)) \supset \forall n X(n)$$

where $PA(X)$ denotes the Peano Arithmetic with an additional unary predicate X. Let $<_{\varepsilon_0}$ denote a standard elementary recursive ε_0 well ordering and ERA the Elementary Recursive Arithmetic.

Theorem 1.1 *If $\prec$ is an irreflexive, transitive and provably well founded (not necessarily a total ordering) relation on ω, then there exists an ordinal $\alpha < \varepsilon_0$ and an elementary recursive function f so that ERA proves that*

$$\forall n(n \not\prec n) \,\&\, \forall n, m, k(n \prec m \prec k \supset n \prec k) \supset$$
$$\forall n, k[(n \prec k \supset f(n) <_{\varepsilon_0} f(k) \,\&\, f(n) <_{\varepsilon_0} \alpha]$$

Proof. Work in ERA. From the proof in [24] pp.149-154 we see that there exist an ordinal $\alpha < \varepsilon_0$ and an elementary recursive function h so that $h(k)$ is an additive principal number$<_{\varepsilon_0} \omega^\alpha$, i.e., $h(k) = \omega^\beta$ for some β and

$$\forall k[\forall n < k(k \prec n \supset h(k) <_{\varepsilon_0} h(n))] \tag{1}$$

($<$ denotes the usual ordering on ω. A definition of the function h will be sketched below.)
Define

$$f(n) = \max_{<_{\varepsilon_0}}\{h(n_0)\# \cdots \# h(n_l) : n_0 \prec \cdots \prec n_l = n \,\&\, n_0, \ldots, n_{l-1} < n\}$$

Here $\max_{<_{\varepsilon_0}}$ denotes the maximum with respect to the ordering $<_{\varepsilon_0}$ and note that $\prec$ is irreflexive. The function f is elementary recursive and we have

Claim 1.1 $n \prec k \Rightarrow f(n) <_{\varepsilon_0} f(k)$

Proof of Claim 1.1. ■ Assume $n \prec k$. Choose a sequence $n_0, \dots, n_l$ so that $f(n) = h(n_0)\#\cdots\#h(n_l)$, $n_0 \prec \cdots \prec n_l = n \,\&\, n_0, \dots, n_{l-1} < n$. By transitivity of $\prec$ we have $n_i \prec k$ for any $i \le l$. Partition the set $\{0, \dots, l\}$ into two sets A and B as follows: $A = \{i \le l : k < n_i\}$, $B = \{i \le l : n_i < k\}$ ($\prec$ is irreflexive.) By (1) we have $h(n_i) <_{\varepsilon_0} h(k)$ for each $i \in A$, and hence $\#\sum\{h(n_i) : i \in A\} <_{\varepsilon_0} h(k)$ since $h(k)$ is additive principal. ($\#\sum\{\alpha_0, \dots, \alpha_n\}$ denotes $\alpha_0\#\cdots\#\alpha_n$.)
On the other hand we have, using the transitivity of $\prec$,

$$\#\sum\{h(n_i) : i \in B\} \le_{\varepsilon_0}$$
$$\max_{<_{\varepsilon_0}}\{h(k_0)\#\cdots\#h(k_{m-1}) : k_0 \prec \cdots \prec k_{m-1} \prec k \,\&\, k_0, \dots, k_{m-1} < k\}$$

Therefore we get $f(n) <_{\varepsilon_0} f(k)$. □

Sketch of a definition of the function h.
We follow notations and terminology in [24].

1) Define a *TJ proof* exactly as in [24], p.149. That is, a TJ proof may have TJ initial sequents as extra initial sequentes:
TJ initial sequent $\forall x \prec t X(x) \rightarrow X(t)$ ($X(t)$ is called the *principal formula* of the TJ initial sequent.). Also a TJ proof ends with a sequent of the form $\rightarrow X(m_0), \dots, X(m_n)$. We identify the mth numeral with the number m.

2) The ordinal assignment $o(P)$ for a TJ proof P is defined as in [24].

3) A TJ proof is called *noncritical* if one of the reduction steps for PA which lowers the ordinal applies to it. Otherwise it is called *critical.*

4) We say that a TJ proof P' is the *noncritical reduct* of a noncritical TJ proof P if P' is obtained from P by applying a reduction step for PA which lowers the ordinal.

5) We call a formula in the end-piece of a TJ proof, a *principal TJ descendent* if it is a descendent of a principal formula of a TJ initial sequent. If P is a critical TJ proof, then the endsequent of P contains a principal TJ descendent (cf. [24], pp.151-152.).

6) Let P be a critical TJ proof of $\rightarrow X(m_0), \dots, X(m_n)$, and k be a number such that $k \prec m_i$ for every $i \le n$. For some $i \le n$ the fomula $X(m_i)$ in the endsequent $\rightarrow X(m_0), \dots, X(m_n)$ is a principal TJ descendent of a TJ initial sequent $\forall x \prec m_i X(x) \rightarrow X(m_i)$. Then add the formula $X(k)$ to the endsequent and replace the TJ initial sequent $\forall x \prec m_i X(x) \rightarrow X(m_i)$ by the following proof:

$$\dfrac{\dfrac{\dfrac{\rightarrow k \prec m_i \quad X(k) \rightarrow X(k)}{k \prec m_i \supset X(k) \rightarrow X(k)}}{\forall x \prec m_i X(x) \rightarrow X(k)}}{\forall x \prec m_i X(x) \rightarrow X(k), X(m_i)}$$

If P' is obtained from a critical P and k in this way, then we say that P' is the *critical reduct of P at k.*

7) Since $\prec$ is provably well founded, we have in the system formed from $PA(X)$ by adjoining TJ initial sequents, a proof $P(a)$ of the sequent $\rightarrow X(a)$ for a free variable a. Then, for each k, $P(k)$ denotes a TJ proof of $\rightarrow X(k)$ obtained from $P(a)$ by substituting the numeral k for the variable a.

8) Now let us define, for each number k, a TJ proof P_k by induction on k so that for every n, if $X(n)$ occurs in the endsequent of P_k, then $k \preceq n$ ($\Leftrightarrow_{df}$ $k \prec n$ or $k = n$).

8.1) The case $\neg\exists n < k(k \prec n)$: Then $P_k = P(k)$. The endsequent of P_k is $\rightarrow X(k)$.

8.2) The case $\exists n < k(k \prec n)$: Pick an $n_0 < k$ so that $k \prec n_0$ and $\forall n < k(k \prec n \Rightarrow o(P_{n_0}) \le_{\varepsilon_0} o(P_n))$.

8.21) If P_{n_0} is noncritical, then P_k is defined to be the noncritical reduct of P_{n_0}.

8.22) If P_{n_0} is critical, then P_k is defined to be the critical reduct of P_{n_0} at k. In **8.21** the endsequent is unchanged, while in **8.22** it is augmented with the formula $X(k)$. In any cases we have $o(P_k) <_{\varepsilon_0} o(P_{n_0})$.

9) Finally we set: $h(k) =_{df} \omega^{o(P_k)}$. Then the required condition (1) is clearly enjoyed.

□

2 Elementary bound for cut elimination in propositional calculus

It is well known that the length of the shortest cut free proof is bouded by an elementary function of the length of the original proof in propositional calculus, e.g., cf.[15]. In this section we give a simple proof of this fact.This yields $S^0_2(X) \neq T^0_2(X)$ as a corollary.
Let LK_0 denote a classical propositional calculus in a Tait calculus. To be definite LK_0 denotes the calculus for the propositional part in [20]. Γ, Δ denotes *sequents*, i.e., finite sets of formulae. (Ax) $\Gamma, \neg A, A$ (for an atomic A) is the only initial sequent in LK_0. Inference rules are $(\wedge)$, $(\vee)$ and (cut). A precise fomulation of these rules is irrelevant to our proof. Each *proof* in LK_0 is a tree of sequents.
For a proof P in LK_0, the *depth* of P, denoted by $dp(P)$, is defined to be the depth of the tree P, i.e., the length of the longest branch in the tree P. The *length* of P, denoted by $lh(P)$, is defined to be the total number of occurrences of inference rules in P. Clearly we have $lh(P) < 2^{dp(P)}$ since each inference rule is at most binary.

Theorem 2.1 *If P_0 is a proof of a sequent Γ_0 in LK_0, then there exists a cut free proof P of Γ_0 so that $dp(P) \leq lh(P_0)$. Therefore $lh(P) < 2^{lh(P_0)}$.*

Proof. First eliminate cuts in the given proof P_0 by a usual cut elimination procedure, e.g., in [20]. The resulting cut free proof is denoted by P^{cf}. We say that two infernce rules J_0 and J_1 are *similar* if 1) these are the same type of rules, e.g., both rules are $(\wedge)$ and 2) their auxiliary formulae and principal formulae are the same. We denote this equivalence relation by $J_0 \simeq J_1$. For example,

$$\frac{\Gamma, A_0 \quad \Gamma, A_1}{\Gamma, A_0 \wedge A_1} J_0 \qquad \frac{\Delta, B_0 \quad \Delta, B_1}{\Delta, B_0 \wedge B_1} J_1$$

$J_0 \simeq J_1 \Leftrightarrow (A_0, A_1) = (B_0, B_1)$.
Then it is obvious that for each inference rule J in P^{cf} there exists a J' in P_0 such that $J \simeq J'$. Hence $k \leq lh(P_0)$ with the maximum number k of equivalence classes of inference rules in a branch in P^{cf}. Thus it suuffices to show that we can collapse two similar inference rules in a branch into a single one. For example if a rule

$$\frac{\Gamma, A_0 \quad \Gamma, A_1}{\Gamma, A_0 \wedge A_1} J_0$$

is above the left uppersequent Δ, A_0 of another rule

$$\frac{\Delta, A_0 \quad \Delta, A_1}{\Delta, A_0 \wedge A_1} J_0$$

then eliminate J_0 to get the sequent $\Gamma, A_0 \wedge A_1, A_0$ and absorb the formula $A_0 \wedge A_1$ into the principal fomula at J_1. In this way we get another cut free proof P such that no branch in P contains a pair of similar inference rules. Therefore $dp(P) \leq lh(P_0)$ as desired. □

Let $S^0_2(X)$ denote a bounded arithmetic obtained from Buss' S^1_2 in [5] by adding a unary predicate X together with the equality axiom for the extra X and replacing $\Sigma^b_1 - PIND$ by $\Sigma^b_0(X) - PIND$. $\Sigma^b_0(X)$ denotes the set of sharply bounded formulae in the language augmented by X. Also $T^0_2(X)$is obtained from $S^0_2(X)$ by replacing $\Sigma^b_0(X) - PIND$ by $\Sigma^b_0(X) - IND$. We show:

Corollary 2.1 *$S^0_2(X) \not\vdash X(0) \wedge \forall x(X(x) \supset X(x+1)) \supset \forall x X(x)$, i.e., $S^0_2(X) \neq T^0_2(X)$.*

Assume that $S^0_2(X) \vdash X(0) \wedge \forall x(X(x) \supset X(x+1)) \supset \forall x X(x)$. Let S denote a system arising from $S^0_2(X)$ such that 1) the language of S is the same as one of $S^0_2(X)$, 2) we add initial sequents $\Gamma, X(0)$ in S and 3) we add an inference rule

$$\frac{\Gamma, X(t)}{\Gamma, X(t+1)} (prg)$$

Then we have $S \vdash X(a)$ for a variable a.
Let T denote a propositional calculus arising from LK_0 such that 1) the atoms in T are X_n $(n \in \omega)$, 2) we add initial sequents Γ, X_0 in T and 3) we add an inference rule (prg_n) for each $n \in \omega$

$$\frac{\Gamma, X_n}{\Gamma, X_{n+1}} (prg_n)$$

For each $\Sigma^b_0(X)$ sentence A a propositional formula A^* is associated as follows: 1) for an atomic A without X, $A^* = X_0$ if A is true, $A^* = \neg X_0$ otherwise. 2) $X(t)^* = X_n$ with the value n of the closed term t. 3) $*$ commutes with any propositional connectives. 4) $(\exists x \leq tA(x))^* = \bigvee\{A(i)^* : i \leq n\}$ with the value n of the

closed term t and similarly for $\forall x \leq t$. For a sequent $\Gamma = \{A_0, \dots, A_m\}$ consisting solely of $\Sigma_0^b(X)$ sentences, we set $\Gamma^* = \{A_0^*, \dots, A_m^*\}$.
Let $\Gamma(\bar{a})$ denote a $\Sigma_0^b(X)$ sequent whose free variables are among the sequence $\bar{a} = (a_0, \dots, a_m)$ of variables. For a sequence $\bar{n} = (n_0, \dots, n_m)$ of natural numbers $\Gamma(\bar{n})$ denotes the result of simultaneous substitution n_i for a_i. Then it is easy to show:

Lemma 2.1 *If $S \vdash \Gamma(\bar{a})$, then there exists a polynomial $f(\bar{a})$ such that for any $\bar{n}$ there exists a proof $P_{\bar{n}}$ of $\Gamma(\bar{n})$ in T such that*

$$lh(P_{\bar{n}}) \leq f(\mid \bar{n} \mid) = f(\mid n_0 \mid, \dots, \mid n_m \mid).$$

($\mid n \mid$ is the length of the binary expansion of the number n.)

This follows from the fact that for each term $t(\bar{a})$ there exists a polynomial g such that $\forall \bar{n}(\mid t(\bar{n}) \mid \leq g(\mid \bar{n} \mid))$. Therefore we would have a polynomial f such that for each n there exists a proof P_n of X_n in T with $lh(P_n) \leq f(\mid n \mid)$. It is fairly easy to extend Theorem 2.1 to the caluculus T. Thus we would have for a polynomial f such that

$$\text{for any } n \text{ there exists a cut free proof } P_n \text{ of } X_n \text{ in } T\, (dp(P_n) < f(\mid n \mid)). \tag{2}$$

We say that a sequent is *positive* if the atom X_n occurrs only positively in it for any n. Put $\vdash^k \Gamma$ iff there exists a cut free proof P of Γ in T such that $dp(P) \leq k$. Also we denote $k \models \Gamma$ if Γ is true under the truth assignment

$$X_n = \text{ if } n \leq k \text{ then } \mathit{true} \text{ else } \mathit{false}.$$

Then for any positive Γ we have $\vdash^k \Gamma \Rightarrow k \models \Gamma$. Now (2) runs $\vdash^{f(|n|)} X_n$ and hence $\forall n(n \leq f(\mid n \mid))$. This is a contradiction.

Remark. Add all polynomial growth rate functions to the language. Denote the set of true Σ_2^0 sentences in this extended language by Tr_{Σ_2}. Let $S_2^0(X) + Tr_{\Sigma_2}$ denote the theory obtained from $S_2^0(X)$ by adding Tr_{Σ_2}. Then we see

$$S_2^0(X) + Tr_{\Sigma_2} \not\vdash X(0) \wedge \forall x(X(x) \supset X(x+1)) \supset \forall x X(x)$$

from the above proof. Observe that $S_2^0(X) + Tr_{\Sigma_2} \vdash \Sigma_\infty^b - IND$ since each instance of $\Sigma_\infty^b - IND$ is in Tr_{Σ_2} for a bounded formula $A \in \Sigma_\infty^b$ without X.

3 Equivalents for Bar Induction

In this section we give some equivalents for Bar Induction.
L_f denotes a second order language containg 1) the language of the first order arithmetic, 2) set variables $X, Y, \dots$ and 3) unary function variables $f, g, \dots$. Σ_0^0 denotes the set of bounded formulae in L_f and Π_0^1 the set of arithmetical (=first order) formulae possibly with second order parameters. We take the theory $\Sigma_0^0 - CA$ as our base theory. The theory $\Sigma_0^0 - CA$ has the following axiom schemata besides the axioms for first order constants:

1. Graph Principle: $\forall x \exists ! y X(j(x,y)) \supset \exists f \forall x X(j(x, fx))$
   (j:a pairing function),
2. Comprehension Axiom for Σ_0^0-formulae and
3. IA:$\forall X[X(0) \,\&\, \forall n\{X(n) \supset X(n+1)\} \supset \forall n X(n)]$

In this section we use signs $\supset$ and $\rightarrow$ interchangeably to denote the propositional connective 'implication'.

Definition 3.1 *1. BI* denotes the axiom schema:
$Hyp1 \,\&\, Hyp2 \,\&\, Hyp3 \supset Q\langle\rangle$ for a $P \in \Sigma_0^0$ and an arbitrary formula Q ($\langle\rangle$ is the empty sequence), where

Hyp1 : $\forall f \exists x P(\bar{f}x)$ $(\bar{f}x = \langle f0, \dots, f(x \dot{-} 1) \rangle)$

Hyp2 : $\forall c \in Seq(Pc \supset Qc)$
(Seq denotes the set of gödel numbers of finite sequences of natural numbers).

Hyp3 : $\forall c \in Seq[\forall x Q(c * \langle x \rangle) \supset Qc]$

2. For a binary relation $\prec$, $Wf(\prec) \Leftrightarrow_{df} \forall f \exists x(f(x+1) \not\prec fx)$

3. $Prg[\prec, Q] \Leftrightarrow_{df} \forall x(\forall y \prec xQy \supset Qx)$
4. $I(\prec, Q) \Leftrightarrow_{df} Prg[\prec, Q] \supset \forall xQx$
5. TI denotes the axiom schema $Wf(\prec) \supset I(\prec, Q)$
   for $\prec \in \Sigma_0^0$ and an arbitrary Q.
6. TI' denotes the axiom schema $\forall XI(\prec, X) \supset I(\prec, Q)$
   for $\prec \in \Sigma_0^0$ and an arbitrary Q.
7. $Ng \Leftrightarrow_{df} \forall f \forall x \exists c \in Seq(lh(c) = x \,\&\, f \in c) \Leftrightarrow_{df} \forall f \forall x \exists c \in Seq(c = \bar{f}x)$
8. For a formula $F(x, y)$, $Fnc(F) \Leftrightarrow_{df} \forall x \exists ! y F(x, y)$ and
   $c = \bar{F}x \Leftrightarrow_{df} lh(c) = x \,\&\, \forall i < xF(i, c(i))$ with the ith component $c(i)$ of the sequence c.
9. NG denotes the axiom schema $Fnc(F) \supset \forall x \exists c(c = \bar{F}x)$
   for an arbitrary F.
10. $\forall E(\Pi_0^1)$ denotes the axiom schema $\forall XA(X) \supset A(\{x\}F(x))$
    for an $A \in \Pi_0^1$ and an arbitrary F.

Theorem 3.1 *(cf. [14]) Over $\Sigma_0^0 - CA$, the following axiom schemata are mutually equivalent:*

$$Ng + BI,\ Ng + TI,\ TI' \text{ and } \forall E(\Pi_0^1)$$

The theorem is seen from a series of the following propositions. Except the direction $Ng + BI \rightarrow \forall E(\Pi_0^1)$ these are due to Howard and Kreisel [14]. Also we learned a weaker result $\Pi_1^0 - CA + BI \rightarrow \forall E(\Pi_0^1)$ from [4], p.52.

Remark. We have also a second order parameter-free version of the theorem.

Proposition 3.1 *1. $Ng + BI \vdash TI$ (cf. [14], Theorem 5A)*

2. $TI \vdash BI$ (cf. [14], Theorem 5C)

3. $TI' \vdash TI$

4. $\forall E(\Pi_0^1) \vdash BI$

5. $TI' \vdash Ng$ and $\forall E(\Pi_0^1) \vdash Ng$

6. $Ng + BI \vdash \Pi_\infty^1 - IA$

7. $Ng + BI \vdash NG$

Proof. 3. It suffices to show, in $\Sigma_0^0 - CA$, $Wf(\prec) \vdash \forall XI(\prec, X)$ for a $\prec \in \Sigma_0^0$. This follows from

$$\forall m(m \notin X \supset \exists n \prec m(n \notin X)) \supset \exists f \forall m(m \notin X \supset fm \prec m \,\&\, m \notin X)$$

4. As in 3, we have

$$Hyp1 \,\&\, Hyp2 \,\&\, Hyp3 \supset X<>$$

for a $P \in \Sigma_0^0$ in $\Sigma_0^0 - CA$. Taking this formula as $A(X)$ in $\forall E(\Pi_0^1)$ we get any instance

$$Hyp1 \,\&\, Hyp2 \,\&\, Hyp3 \supset Q<>$$

of BI.
5. This follows from TI', $\forall E(\Pi_0^1) \vdash \Pi_\infty^1 - IA$.
6. $A(0) \,\&\, \forall n(A(n) \supset A(n+1))$ we have to show $A(a)$. Put $Pc \equiv a \leq lh(c)$ and $Qc \equiv A(a \dot{-} lh(c))$. By Ng we have $Hyp1$. By BI we conclude $Q<>$, i.e., $A(a)$.
7. This follows from 6. □

A formula $A(\vec{f})$ $(\vec{f} = f_0, \ldots, f_n)$ is said to be in $\vec{f}$ *normal form* if each function variable f_i, $i \leq n$ occurs only of the form $f_i(y) = z$ for some variables y, z in the formula $A(\vec{f})$.
In a canonical way, each quantifier free formula $R(\vec{f})$ is transformed into its $\vec{f}$ *normal form* $\exists \vec{x} R_0(\vec{x}, \vec{f})$ with new variables $\vec{x}$ and a quantifier free R_0.
Let $R(f)$ be a quantifier free formula and $F \equiv \{x, y\}F(x, y)$ be a binary formula (abstract). Also let $\exists \vec{x} R_0(\vec{x}, f)$ denote the f normal form of $R(f)$. Then $R(F)$ denotes the result of replacing each $f(x) = y$ in $\exists \vec{x} R_0(\vec{x}, f)$ by $F(x, y)$.

Proposition 3.2 *1. For each quantifier free and f normal form $R(x,f)$ there exists a $P \in \Sigma^0_0$ such that:*

(a) every free variable occurring in P is either a new number variable c or a variable occurring in $R(x,f)$ except x, f.

(b) for any binary formula F,

$$NG \vdash Fnc(F) \to [\exists x R(x,F) \leftrightarrow \exists c(F \in c \,\&\, Pc)]$$

with $F \in c \Leftrightarrow_{df} c = \bar{F}(lh(c))$.

2. For a quantifier free and f normal form $R(x,f)$, and a binary formula F,

$$Ng + BI \vdash Fnc(F) \,\&\, \forall f \exists x R(x,f) \to \exists x R(x,F)$$

3. For a quantifier free $R(x,f)$ and a binary formula F,

$$Ng + BI \vdash Fnc(F) \,\&\, \forall f \exists x R(x,f) \to \exists x R(x,F)$$

4. For a fomula $A(x,y,X)$ let F denote the binary formula:

$$F \equiv F(X) =_{df} \{x,y\}(y \simeq \mu y.\neg A(x,y,X))$$

with $y \simeq \mu y.\neg A \Leftrightarrow_{df} [\exists y \neg A \,\&\, y = \min\{y : \neg A\}] \vee [\forall y A \,\&\, y = 0]$
Then for any formula V,

$$\begin{array}{lll} Ng + BI & \vdash & Fnc(F(V)) \text{ and} \\ Ng + BI & \vdash & \forall x \exists y \neg A(x,y,V) \to \forall x \forall y [F(V)(x,y) \to \neg A(x,y,V)] \\ \text{that is,} & & \\ Ng + BI & \vdash & \exists x A(x, F(V)(x), V) \to \exists x \forall y A(x,y,V) \end{array}$$

Proof.

1. Let $P_0 c$ denote a formula obtained from R by replacing each $f(y) = z$ $[f(y) \neq z]$ by $c(y) = z \,\&\, y < lh(c)$ $[c(y) \neq z \,\&\, y < lh(c)]$, resp. Then put $Pc \Leftrightarrow_{df} \exists x < lh(c) P_0 c$. We need NG to show that $Fnc(F) \,\&\, \exists x R(x,F) \to \exists c(F \in c \,\&\, Pc)$.

2. Assume $Fnc(F) \,\&\, \forall f \exists x R(x,f)$. Let P denote the formula formed in 1. Then we have $Hyp1$ for this P. Put
$$Qc \Leftrightarrow_{df} F \in c \to \exists d(F \in c * d \,\&\, P(c * d))$$
By $F(lh(c),x) \,\&\, Q(c* < x >) \to Qc$, we have $Hyp2 \,\&\, Hyp3$. Thus by BI $Q<>$ and hence $\exists d(F \in d \,\&\, Pd)$. The assertion follows from 1.

3. This follows from Proposition 3.2.2 and the definition of $\exists x R(x,F)$.

4. This follows from Proposition 3.1.6.

□

Lemma 3.1 $Ng + BI \vdash \forall E(\Pi^1_0)$

Proof For an $A \in \Pi^1_0$ and an arbitrary V we have to show $\forall X A(X) \to A(V)$.

Step1 First transform A into a prenex normal form whose leading quantifier is $\exists$. For example assume $A(X) \leftrightarrow \exists x_0 \forall y_0 \exists x_1 \forall y_1 A_0(X)$ with a quantifier free A_0. We need only logical axioms to obtain this equivalence. Hence for any formula V we have $A(V) \leftrightarrow \exists x_0 \forall y_0 \exists x_1 \forall y_1 A_0(V)$. Thus we can assume that A is in prenex normal form, e.g., of the form $\exists x_0 \forall y_0 \exists x_1 \forall y_1 A_0(X)$.

Step2 Second transform A into its Herbrand normal form. Pick new function variables f_0, f_1 and put $A_H \equiv A_0(x_0, f_0(x_0), x_1, f_1(x_0,x_1))$. We have logically $\exists x_0 \forall y_0 \exists x_1 \forall y_1 A_0 \to \forall f_0 \forall f_1 \exists x_0 \exists x_1 A_H$. Put

$$\begin{array}{lll} F_0 & =_{df} & \{x_0, y_0\}(y_0 \simeq \mu y_0.\neg \exists x_1 \forall y_1 A_0) \\ F_1 & =_{df} & \{x_0, x_1, y_1\}(y_1 \simeq \mu y_1.\exists y_0(F_0(x_0,y_0) \,\&\, \neg A_0)) \end{array}$$

By Proposition 3.2.4, $Ng + BI$ proves that

$$Fnc(F_0(V)) \,\&\, Fnc(F_1(V))$$

and

$$\exists x_0 \exists x_1 A_0(x_0, F_0(V)(x_0), x_1, F_1(V)(x_0, x_1), V) \to \exists x_0 \forall y_0 \exists x_1 \forall y_1 A_0(V)$$

Hence, in $Ng + BI$, $\forall X A(X) \to \forall X \forall f_0 \forall f_1 \exists x_0 \exists x_1 A_H$ and $\forall X \forall f_0 \forall f_1 \exists x_0 \exists x_1 A_H \to \exists x_0 \exists x_1 A_0(x_0, F_0(V)(x_0), x_1, F_1(V)(x_0, x_1), V)$. Thus Proposition 3.2.3 yields $\exists x_0 \forall y_0 \exists x_1 \forall y_1 A_0(V) \equiv A(V)$.

□

Now Theorem 3.1 has been proved from these propotions and lemma.

Next we show that Bar Induction is equivalent to the reflection schema for ω logic.
We change our language L_f to L_2:From L_f 1) remove the function variables, 2) add the n-ary predicate variables X_i^n ($i \in \omega$) and 3) restrict function constants to $0, S$ (S:successor). The resulting language is denoted L_2. Thus closed terms in L_2 are numerals. We understand that predicate constants corresponding to primitive recursive relations are included in L_2.
Let LK_2 denote a Tait's calculus for this second order language L_2. A second order terms is just a predicate variable X^n and hence in LK_2 the inference rule ($\exists_2$) for the second order existential quantifier runs:

$$\frac{\Gamma, F(X)}{\Gamma, \exists Y F(Y)}$$

Also let RFN denote the reflection schema for the calculus LK_2.

Proposition 3.3

$$\Sigma_0^0 - CA \vdash RFN \leftrightarrow \Pi_\infty^1 - IA$$

Proof. ($\to$) For each n we have $LK_2 \vdash A(0) \wedge \forall x(A(x) \supset A(Sx)) \supset A(n)$.
($\leftarrow$) By cut elimination and the partial truth definition. □

Let ACA_0 denote the second order arithmetic Arithmetical Comprehension Axiom with Restricted induction. Since ACA_0 is finitely axiomatizable, we get the

Corollary 3.1 *(cf. [22], Lemma 2.7)*

$$ACA_0 \vdash RFN_{ACA_0} \leftrightarrow \Pi_\infty^1 - IA$$

Let $X_0 \subseteq \omega \times \omega$ be a binary relation. $LK_\omega(X_0)$ denotes the ω logic with the relation X_0:

1. The language of $LK_\omega(X_0)$ is obtained from L_2 by adding the binary predicate constant X_0 and removing the first order free variables. Any sequents in $LK_\omega(X_0)$ have no first order free variable.

2. Axioms (=initial sequent) in $LK_\omega(X_0)$ are diagrams for the relation X_0 besides the usual axioms for the constants and logical ones.

$$\Gamma, X_0(n,m) \text{ if } X_0(n,m) \quad \text{, and } \Gamma, \neg X_0(n,m) \text{ if } \neg X_0(n,m)$$

3. Inference rules in $LK_\omega(X_0)$ are those of LK_2 except the following changes. First replacing the usual rule for first order universal quantifier by the ω rule:

$$\frac{\{\Gamma, A(n) : n \in \omega\}}{\Gamma, \forall x A(x)} \ (\omega)$$

Second restrict the rule ($\exists$) for the first order existential quantifier to:

$$\frac{\Gamma, A(n)}{\Gamma, \exists x A(x)}$$

By a *preproof* we mean an ω-branching labelled tree of sequents and some data. Data should include names of axioms or inference rules and finite cut degrees. A preproof have to be locally correct with respect to the data. An *ω-proof* is a well founded preproof.
$\omega - RFN$ denotes the schema saying that if a sequent Γ has an ω-proof in $LK_\omega(X_0)$, then the sequent Γ is true.

Theorem 3.2

$$\Sigma^0_0 - CA_0 \vdash \omega - RFN \leftrightarrow Ng + TI$$

Proof. ($\rightarrow$) We have $\Pi^1_\infty - IA$ and hence Ng. Assume $Wf(\prec)$. We have to show $Prg[\prec, A] \rightarrow \forall x A(x)$. It suffices to show that there exists an ω-proof of the sequent $\neg Prg[\prec, A], \forall x A(x)$. Wlog we can assume that no second order free variable (except the 'constant' X_0) occurrs in $\prec \in \Sigma^0_0$. Thus $n \prec m$ has an ω-proof when $n \prec m$ is true and similarly for the case $n \not\prec m$. Using this fact we can construct a preproof of the sequent $\neg Prg[\prec, A], \forall x A(x)$ which must be well founded bt our assumption $Wf(\prec)$.
($\leftarrow$) Again by cut elimination and the partial truth definition. Here cuts are eliminated through Mints' continuous cut elimination procedure in Mints [18]. To ensure that the resulting cut free preproof is well founded we need $\Pi^0_1 - CA$, i.e., Arithmetical Comprehension Axiom (cf. [18] or [11]). But $\Pi^0_1 - CA$ follows from $Ng + TI \leftrightarrow \forall E(\Pi^1_0)$. □

$\omega - RFN$ is also equivalent to the so called ω-model reflection schema over ACA_0.
Let T be a theory in L_2. By $\omega - model - RFN_T$ we mean the following schema for an arbitrary formula $A(X)$:

$$A(X) \rightarrow \exists \text{ countable } M = (M_n : n \in \omega)(M_0 = X \,\&\, M \models T \,\&\, M \models A[X])$$

When T consists solely of axioms for constants in L_s, we denote simply
$\omega - model - RFN$.

Proposition 3.4 *(Henkin-Orey's ω-completeness theorem) In ACA_0,*
there exists an ω-proof of $A(X_0)$ in $LK_\omega(X_0)$
iff
for any countable ω model $M \ni X_0$ $M \models A[X_0]$

Corollary 3.2 *$ACA_0 \vdash \omega - RFN \leftrightarrow \omega - model - RFN$ and hence*
$ACA_0 \vdash \omega - model - RFN_{ACA_0} \leftrightarrow TI'$ (cf. [22].)

4 Direct computations in an equatinal calculus PRE

Let PRE denote the theory PRA *minus induction axiom.* The axioms of PRE are defining equations for primitive recursive functions: 0 (zero) denotes an invividual constant ans S the successor function. Function constants and their defining equations are generated as follows.

1. (projection) $I^n_i(x_1, \ldots, x_n) = x_i$ $(1 \leq i \leq n, n > 0)$
2. (composition) $f(\bar{x}) = h(g_1(\bar{x}), \ldots, g_m(\bar{x}))$,
   where $\bar{x}$ denotes a sequence $x_1, \ldots, x_n$ of variables.
3. (primitive recursion 1) $f(0) = k,\ f(Sy) = h(y, f(y))$
   (k is a natural number).
4. (primitive recursion 2) $f(\bar{x}, 0) = g(\bar{x}),\ f(\bar{x}, Sy) = h(\bar{x}, y, f(\bar{x}, y))$

In what follows our concern is restricted to Horn clauses $E \supset e$ with an equation e and a finite set E of equations. Therefore it is better to consider PRE as an equational theory with an extra axiom E. By $E \vdash e$ we mean the equation e is derivable from the set E in PRE.
For a function constant f, $Cl(f)$ denotes the finite set of function constants which are used to define the constant f (and the successor S). Specifically,

1. (projection) $Cl(I^n_i) = \{I^n_i\} \cup \{S\}$
2. (composition) $Cl(f) = Cl(h) \cup \bigcup\{Cl(g_i) : 1 \leq i \leq m\} \cup \{f\}$
3. (primitive recursion 1) $Cl(f) = Cl(h) \cup \{f\}$
4. (primitive recursion 2) $Cl(f) = Cl(g) \cup Cl(h) \cup \{f\}$

For a term t, $Cl(t) = \bigcup\{Cl(f) : f \text{ occurs in } t\}$.
For an equation $t = s$, $Cl(t = s) = Cl(t) \cup Cl(s)$.
For a set E of equations, $Cl(E) = \bigcup\{Cl(e) : e \in E\}$.

Let R denote the set of rules $l \to r$ which are obtained from one of the defining equations $l = r$ by replacing the equality sign $=$ by the arrow $\to$. Viewing the set R as a term-rewriting system, $s \to_R t$ or simply $s \to t$ denotes the relation "the term s rewrites to the term t by R" in the sense of [7]. Also $\xrightarrow{*}$ denotes the reflexive-transitive closure of $\to$ and $\xleftrightarrow{*}$ the smallest congruent relation containg the relation R.
The following is a folklore.

Proposition 4.1 *1. $\to$ is Church-Rosser, i.e., $\xleftrightarrow{*} \subseteq \xrightarrow{*} \circ \xleftarrow{*}$*

2. $\to$ is terminating.

3. $s \to t \Rightarrow Cl(s) \subseteq Cl(t)$

Therefore we have

Proposition 4.2 *For an equation e, $\vdash e$ is decidable by computing the normal forms of the both sides of e.*

Proposition 4.3 *For an equation e, if $\vdash e$, then there exists a direct (in the sense of [21],p.343) computation $\mathcal{D}$ of e; every function constant occurring in $\mathcal{D}$ is in $Cl(e)$.*

This directness does not hold if we replace $\vdash e$ by $E \vdash e$.
Counterexamples (H. Friedman [9])

1. $Sx = Sy \vdash x = y$: Apply the predecessor function pd.

2. $0 = Sx \vdash y = z$: Apply the descriminator δ, $\delta(y, z, 0) = y$, $\delta(y, z, Sx) = z$.

Our theorem below says that these are only exceptions.

Definition 4.1 PRE' is obtained from PRE by adding the following two rules for an arbitrary equation e:

$$\frac{St_1 = St_2}{t_1 = t_2}\,(S) \qquad \frac{0 = St}{e}\,(\delta)$$

Theorem 4.1 *For a finite set E of equations and an equation e, if $E \vdash e$, then there exists a direct computation $\mathcal{D}$ of e from E in PRE';every function constant occurring in $\mathcal{D}$ is in $Cl(E) \cup Cl(e)$.*

Corollary 4.1 *For an open formula A, if $\vdash A$, then there exists a direct derivation $\mathcal{D}'$ of A in PRE' and hence a weakly direct derivation $\mathcal{D}$ of A in PRE; every function constant occurring in $\mathcal{D}'$ [in $\mathcal{D}$] is in $Cl(A)$ [$Cl(A) \cup Cl(pd) \cup Cl(\delta)$], resp.*

Proof of Corollary. Write A in CNF $\bigwedge\{C : C \in \Gamma\}$ and consider each conjunct C separately. C is equivalent to $E' \to E$ for some finite sets E' and E of equations ($E' \to E$ denotes a sequent in Gentzen's sense). Then use the theorem and the fact: $E' \vdash E \Rightarrow E' \vdash e$ for some $e \in E$. □

We don't know an answer to the problem raised by H. Friedman [9].
Problem.

1. Is $\vdash A$ decidable for an open A?

2. Is $e \vdash$ decidable for an equation e?

But we conjecture the following.
Conjecture. Let t_1 and t_2 be normal terms with respect to $\to_R$. Then
$t_1 = t_2 \vdash \Leftrightarrow t_1 \equiv S^m t_0$ for some term t_0 and $t_2 \equiv S^n 0$ with $m > n$ or vice versa.

This means that the theory PRE can discriminate between terms only when one is 0 and the other is of the form St. Unless the equation $t_1 = t_2$ is of the form $St = u$ where the term u occurs in t and u contains a variable, the conjecture is easily seen to hold. Also if u is a variable x, then, by H. Friedman [9], we have $St(x) = x \not\vdash$. That's all what we know about the conjecture.

The rest of the section is devoted to a proof of the Theorem 4.1. Fix a finite set E of equations. $t_1 \doteq t_2$ denotes ambiguously the equation $t_1 = t_2$ or $t_2 = t_1$.

Definition 4.2 *1.* $d(E)$ denotes the smallest set of equations such that *1)* $E \subseteq d(E)$ and *2)* $d(E)$ is closed under the rules (S), (sub) and (red):

$$\frac{e[t_1/x] \quad t_1 \dot{=} t_2}{e[t_2/x]}\ (sub) \qquad \frac{t \dot{=} u}{t' \dot{=} u}\ (red) \quad \text{where } t \to_R t'$$

2. $t \to_E t' \Leftrightarrow_{df}$ there exists a term t_0 and a finite set $\{u_i \dot{=} v_i : i < n\} \subseteq d(E)\,(n \geq 0)$ of equations such that $t \equiv t_0[u_0, \ldots, u_{n-1}/x_0, \ldots, x_{n-1}]$ (simultaneous substitution) and $t_0[v_0, \ldots, v_{n-1}/x_0, \ldots, x_{n-1}] \stackrel{=}{\to}_R t'$ ($\stackrel{=}{\to}_R$ denotes the reflexive closure of $\to_R$.)

3. $\stackrel{*}{\to}_E$ $[\stackrel{*}{\leftrightarrow}_E]$ denotes the reflexive-transitive [-symmetric] closure of $\to_E$, resp.

Clearly $e \in d(E) \Rightarrow Cl(e) \subseteq Cl(E)$ and hence we have the

Proposition 4.4 *1.* $t \to_E t' \Rightarrow Cl(t') \subseteq Cl(t) \cup Cl(E)$

2. $E \vdash t_1 = t_2 \Rightarrow t_1 \stackrel{*}{\leftrightarrow}_E t_2$

Lemma 4.1 *Assume that* $0 = St \notin d(E)$ *for any term* t*. Then* $\to_E$ *is Curch-Rosser, i.e.,* $\stackrel{*}{\leftrightarrow}_E \subseteq \stackrel{*}{\to}_E \circ \stackrel{*}{\leftarrow}_E$.

Proof of Theorem 4.1. If $0 = St \notin d(E)$ for any t, then we get the theorem by the Lemma 4.1 and the Proposition 4.4. Assume $0 = St \in d(E)$ for some term t. Then there exists a direct computation $\mathcal{D}$ of $0 = St$ from E in PRE'. By adjoining the rule (δ) we get a desired computation of $t_1 = t_2$ from E. □

In what follows we assume that $0 = St \notin d(E)$ for any term t.

Definition 4.3 *1.* $t \to_I s$ is defined inductively as follws:

(a) $t \to_I t$

(b) $\bar{t} \to_I \bar{s} \Rightarrow f(\bar{t}) \to_I f(\bar{s})$ where, for sequences $\bar{t} \equiv t_1, \ldots, t_n$, $\bar{s} \equiv s_1 \ldots, s_n$ of terms, $\bar{t} \to_I \bar{s} \Leftrightarrow_{df}$ $t_i \to_I s_i$ for any i.

(c) (projection) $t_i \to_i s \Rightarrow I^n_i(t_1, \ldots, t_n) \to_I s$

(d) (composition) $\bar{t} \to_I \bar{s} \Rightarrow f(\bar{t}) \to_I h(g_1(\bar{s}), \ldots, g_m(\bar{s}))$

(e) (primitive recursion 1) $f(0) \to_I k$; $t \to_I u \Rightarrow f(St) \to_I h(u, f(u))$ if $f(0) = k$.

(f) (primitive recursition 2) $\bar{t} \to_I \bar{s} \Rightarrow f(\bar{t}, 0) \to_I g(\bar{s})$;
$\bar{t} \to_I \bar{s} \,\&\, u \to_I v \Rightarrow f(\bar{t}, Su) \to_I h(\bar{s}, v, f(\bar{s}, v))$

2. $t \to_{EI} s \Leftrightarrow_{df}$ there exist a term t_0 and a sequence $\{u_i \dot{=} v_i : i < n\} \subseteq d(E)$ such that $t \equiv t_0[u_0, \ldots, u_{n-1}/x_0, \ldots, x_{n-1}] \to s$.

As usual we have

1. $\stackrel{*}{\to}_R = \stackrel{*}{\to}_I$
2. $\stackrel{*}{\to}_E = \stackrel{*}{\to}_{EI}$
3. $\to_I$ is strongly confluent, i.e., satisfies the diamond property:
$\forall t, s, u \exists v (t \to_I s \,\&\, t \to_I u \Rightarrow s \to_I v \,\&\, u \to_I v)$

Thus it suffices to show the following lemma.

Lemma 4.2 $\to_{EI}$ *is strongly confluent.*

Define

$t_1 \leftrightarrow_{CE} t_2 \Leftrightarrow$
there exist a term t_0 and a sequence $\{u_i \dot{=} v_i : i < n\} \subseteq d(E)$ such that
$t_1 \equiv t_0[u_0, \ldots, u_{n-1}/x_0, \ldots, x_{n-1}]$ and $t_2 \equiv t_0[v_0, \ldots, v_{n-1}/x_0, \ldots, x_{n-1}]$.

Since $d(E)$ is closed under the rule, (sub) $\leftrightarrow_{CE}$ is transitive. Therefore it suffices to show:

Claim 4.1 *If we have* $M_1 \leftarrow_I M \leftrightarrow_{CE} N \to_I N_1$*, then there exist terms* N_2, M_2, L *such that* $M_1 \leftrightarrow_{CE} N_2 \to_I L \leftarrow_I M_2 \leftrightarrow_{CE} N_1$.

Proof of Claim 4.1. We prove this by induction on $m+n$, where m $[n]$ denotes the depth of a derivation of $M \to_I M_1$ $[N \to_I N_1]$, resp.

Case 0 $M \dot{=} N \in d(E)$: Then $M_1 \dot{=} N_1 \in d(E)$. Take $N_2 \equiv L \equiv M_2 \equiv N_1$.

Case 1 $M_1 \equiv M$: Take $N_2 \equiv N$, $L \equiv M_2 \equiv N_1$.

Case 2 $M \equiv f(\bar{t}) \to_I f(\bar{u}) \equiv M_1$ with $\bar{t} \to_I \bar{u}$:

2.1 $N \equiv f(\bar{v}) \to_I f(\bar{w}) \equiv N_1$ with $\bar{v} \to_I \bar{w}$: For each i we have $u_i \leftarrow_I t_i \leftrightarrow_{CE} v_i \to_I w_i$. By IH $u_i \leftrightarrow_{CE} v_i' \to_I s_i \leftarrow_I t_i' \leftrightarrow_{CE} w_i$ for some v_i', s_i, t_i'.

2.2 $N \equiv I_i^n(\bar{v}) \to_I w_i \equiv N_1$ with $v_i \to_I w_i$: As in 2.1, $M_1 \equiv I_i^n(\bar{v}) \leftrightarrow_{CE} I_i^n(u_1, \ldots, u_{i-1}, v_i', u_{i+1}, \ldots, u_n) \to_I s_i \leftarrow_I t_i' \leftrightarrow_{CE} w_i$ for some v_i', s_i, t_i'.

2.3 $N \equiv f(\tilde{v}, 0) \to_I g(\tilde{w}) \equiv N_1$ with $\tilde{v} \to \tilde{w}$ $(\tilde{v} \equiv v_1, \ldots, v_{n-1})$: By IH pick v_i', s_i, t_i' for $i \neq n$ as in 2.1. Then $M_1 \equiv f(\tilde{u}, u_n) \leftrightarrow_{CE} f(\tilde{v}', 0) \to_I g(\tilde{s}) \leftarrow_I g(\tilde{t}') \leftrightarrow_{CE} g(\tilde{w}) \equiv N_1$ by $0 \dot{=} t_n \in d(E) \,\&\, t_n \to_I u_n \Rightarrow 0 \dot{=} u_n \in d(E)$

2.4 $N \equiv f(\tilde{v}, Sv_n) \to_I h(\tilde{w}, w_n, f(\tilde{w}, w_n)) \equiv N_1$ with $\tilde{v} \to_I \tilde{w}$ and $v_n \to_I w_n$: Pick $\tilde{v}'.\tilde{t}', \tilde{s}$ so that $\tilde{u} \leftrightarrow_{CE} \tilde{v} \to_I \tilde{s} \leftarrow_I \tilde{t}' \leftrightarrow_{CE} \tilde{w}$.

2.41 $t_n \dot{=} Sv_n \in d(E)$: Then $u_n \dot{=} Sw_n \in d(E)$. $M_1 \equiv f(\tilde{u}, u_n) \leftrightarrow_{CE} f(\tilde{v}', Sw_n) \to_I h(\tilde{s}, w_n, f(\tilde{s}, w_n)) \leftarrow_I h(\tilde{t}', w_n, f(\tilde{t}', w_n)) \leftrightarrow_{CE} h(\tilde{w}, w_n, f(\tilde{w}, w_n, f(\tilde{w}, w_n)) \equiv N_1$

2.42 Otherwise: $t_n \equiv St$ with $t \leftrightarrow_{CE} v_n$ for some t. Also, for some u, $t \to_I u$ by a shoter or equal length derivation that $t_n \equiv St \to_I Su \equiv u_n$. By IH pick v', s, t' so that $u \leftrightarrow_{CE} v' \to_I s \leftarrow_I t' \leftrightarrow_{CE} w_n$. Then $M_1 \equiv f(\tilde{u}, u_n) \leftrightarrow_{CE} f(\tilde{v}', Sv') \to_I h(\tilde{s}, s, f(\tilde{s}, s)) \leftarrow_I h(\tilde{t}', t', f(\tilde{t}', t')) \leftrightarrow_{CE} h(\tilde{w}, w_n, f(\tilde{w}, w_n)) \equiv N_1$

2.5 $N \equiv f(0) \to_I k \equiv N_1$: Similar to 2.3.

2.6 $N \equiv f(Sv) \to_I h(w, f(w))$ with $v \to_I w$: Similar to 2.4.

Case3 $M \equiv I_i^n(\bar{t}) \to_I u_i \equiv M_1$ with $t_i \to_I u_i$ and $N \equiv I_i^n(\bar{v}) \to_I w_i \equiv N_1$ with $v_i \to_I w_i$

Case 4 $M \equiv f(\bar{t}, 0) \to_I g(\bar{u})$ with $\bar{t} \to_I \bar{u}$: $N \equiv f(\bar{v}, v)$ with $0 \leftrightarrow_{CE} v$, i.e., $0 \dot{=} v \in d(E)$ or $v \equiv 0$. By our assumption, $v \not\equiv Sv'$ for any v'. Therefore it must be the case $v \equiv 0$ and $N \equiv f(\bar{v}, 0) \to_I g(\bar{w})$ with $\bar{v} \to_I \bar{w}$. Use IH.

Case 5 $M \equiv f(\bar{t}, St) \to_I h(\bar{u}, u, f(\bar{u}, u))$ with $\bar{t} \to_I \bar{u}$ and $t \to_I u$: As in the Case 4 we have $N \equiv f(\bar{v}, Sv) \to_I h(\bar{w}, w, f(\bar{w}, w))$ with $\bar{w} \to_I \bar{w}$ and $v \to_I w$. Note that if $St \leftrightarrow_{CE} Sv$, then $t \leftrightarrow_{CE} v$ by the rule (S).

Case 6 $M \equiv f(0) \to_i k$: Similar to the Case 4.

Case 7 $M \equiv f(St) \to_I h(u, f(u))$ with $t \to_I u$: Similar to the Case 5.

This completes a proof of the Claim 4.1.

5 Intuitionistic fixed point theories

In [3] Buchholz shows that an intuitionistic fixed point theory $\hat{ID}_1^i$ is conservative over Heyting Arithmetic HA with respect to almost negative formulae. The proof in [3] is based on a recursive realizability interpretation of the theory $\hat{ID}_1^i$. Having seen a preliminary version of [3] we can extend and strengthen this result.

Our proof runs as follows. First an extension of an intuitionistic iterated fixed point theory $\hat{ID}_n^i$ is interpreted in the intuitionistic analysis $EL + AC - NF$. This is done by imitating Aczel's proof in [8] which shows that the classical fixed point theory $\hat{ID}_1$ is interpretable in a second order arithmetic $\Sigma_1^1 - AC$. Then by N. Goodman's theorem [12] one can conclude our theorem. A proof of N. Goodman's theorem is based on either a combination of a realizability interpretation and a forcing or a proof theoretic analysis in G. Mints [18]. It seems that a direct analysis of $\hat{ID}_n^i$ based on one of thses methods is desirable.

Definition 5.1 *1.* EL denotes the intuitionistic elementary analysis defined in *[25] p.144*. Function variables in EL are denoted by $\alpha, \beta, \gamma, \ldots$

2. The axiom schema $AC-NF$: $\forall n\exists\alpha A(n,\alpha)\supset\exists\beta\forall nA(n,(\beta)_n)$ with $(\beta)_n=\lambda m\beta(j(n,m))$ and a pairing function j.

3. L denotes the language of EL. For a list of set parameters $\bar{X}=X_0,X_1,\ldots$, $L(\bar{X})$ denotes the expanded language obtained from L by adding $\bar{X}$.

4. $EL(\bar{X})$ [$EL+AC-NF(\bar{X})$] denotes the extension of EL [$EL+AC-NF$] by expanding the language to $L(\bar{X})$, resp.
Each axiom schema in EL [$EL+AC-NF$] is available for $L(\bar{X})$ formulae in $EL(\bar{X})$ [$EL+AC-NF(\bar{X})$], resp.

Lemma 5.1 *For each n and each list $\bar{X}$ of set parameters there exists a formula $S_n(x_0,x_1,\ldots,x_n;\bar{X},\alpha)$ in $\Sigma^0_1(x_0,x_1,\ldots,x_n;\bar{X},\alpha)$ such that for every formula A in $\Sigma^0_1(x_0,x_1,\ldots,x_n;\bar{X},\alpha)$ there is an integer e such that*

$$EL(\bar{X})\vdash A\leftrightarrow\Sigma^0_1(e,x_1,\ldots,x_n;\bar{X},\alpha)$$

Proof. By formalizing the enumeration theorem. This is done in $EL(\bar{X})$. cf. Ch. 3, Sect. 6 and 7 in [25]. □

Definition 5.2 Let $\bar{Y}$ be a list of set parameters and $\mathcal{F}$ a set of formulae in $L(\bar{Y})$. Pick an $X\notin L(\bar{Y})$.

1. $POS(\mathcal{F};\bar{Y})=_{df}$the set of all $L(\bar{Y},X)$ formulae which are built up from formulae $X(t)$ (t: a term) and formulae in $\mathcal{F}$ by means of $\wedge,\vee,\forall m,\exists m$ (first order quantifications).

2. $POS^*(\mathcal{F};\bar{Y})=_{df}\{\Phi\in POS(\mathcal{F};\bar{Y}):FV(\Phi)\subseteq\{x\}\}$ for a fixed number variable x. $FV(\Phi)$ denotes the set of free variables occurring in Φ. Thus no function free variable occurs in $\Phi\in POS^*(\mathcal{F};\bar{Y})$.

3. $POS(\bar{Y})=POS(\mathcal{F}_{\bar{Y}};\bar{Y})$ and $POS^*(\bar{Y})=POS^*(\mathcal{F}_{\bar{Y}};\bar{Y})$ with the set $\mathcal{F}_{\bar{Y}}$ of all formulae in $L(\bar{Y})$.

4. $POS(\bar{Y};\bar{Y})=POS(\mathcal{A}_{\bar{Y}};\bar{Y})$ and $POS^*(\bar{Y};\bar{Y})=POS^*(\mathcal{A}_{\bar{Y}};\bar{Y})$ with the set $\mathcal{A}_{\bar{Y}}$ of atomic formulae $Y_i(t)$ for $Y_i\in\bar{Y}$.

5. $POS=POS(\emptyset)$.

Remark. $POS(\mathcal{F};\bar{Y})$ is narrower than strictly positive formulae (with respect to X) because $A\supset X(t)\notin POS(\mathcal{F};\bar{Y})$ but is wider than POS in [3]. If we set $A\supset X(t)\in POS(\mathcal{F};\bar{Y})$, then one would need IP (Independence of Premise) for a proof of Lemma 5.3 below.

Lemma 5.2 *For each $\Phi\in POS$ there exist a list $\bar{Y}$ of set parameters, a $\Phi'\in POS(\bar{Y};\bar{Y})$ and a list $\bar{A}$ of frmulae in L such that*

$$EL(X)\vdash\Phi\leftrightarrow\Phi'[\bar{A}/\bar{Y}]$$

where $[\bar{A}/\bar{Y}]$ denotes the simultaneous substitution.

A formula in $L(\bar{Y})$ is said to be an $n-\Sigma^1_1(\bar{Y})$ *fomula* (Σ^1_1 formula in normal form with set parameters $\bar{Y}$) if it is of the form $\exists\alpha\forall nR(\alpha,n,\bar{Y})$ with an open formula R in $L(\bar{Y})$ in which no function variable except α occurs.

Lemma 5.3 *For each $\Phi\in POS(\bar{Y};\bar{Y})$ and each $A(x)$ in $n-\Sigma^1_1(\bar{Y})$ there exists a C in $n-\Sigma^1_1(\bar{Y})$ such that*

$$EL+AC-NF(\bar{Y})\vdash\Phi[A/X]\leftrightarrow C$$

Proof by induction on the length of Φ using the facts:

$$EL(\bar{Y})\vdash A\vee\exists\alpha B\leftrightarrow\exists\alpha(A\vee B)$$

$$EL(\bar{Y})\vdash\forall zA\vee\forall yB\leftrightarrow\exists x\forall z\forall y[(x=0\wedge A)\vee(x\neq0\wedge B)]$$

□

Lemma 5.4 *For each $\Phi\in POS^*(\bar{Y};\bar{Y})$ there exists a formula $P^\Phi(\bar{Y},x)$ in $n-\Sigma^1_1(\bar{Y})$ such that*

$$EL+AC-NF(\bar{Y})\vdash\forall x\{P^\Phi(\bar{Y},x)\leftrightarrow\Phi[\{x\}P^\Phi(\bar{Y},x)/X]\}$$

Proof by Lemmata 5.1 and 5.3. Put $B(u,x;\bar{Y}) \equiv \exists\alpha\forall y S_3(u,u,y,x;\bar{Y},\alpha)$. Pick an $n-\Sigma^1_1(\bar{Y})$ formula $C \equiv \exists\alpha\forall y C_0(u,y,x;\bar{Y},\alpha)$ such that $\Phi[\{x\}B/X] \leftrightarrow C$. Pick an e so that $C_0(u,y,x;\bar{Y},\alpha) \leftrightarrow S_3(e,u,y,x;\bar{Y},\alpha)$. Then $P^\Phi(\bar{Y},x) \equiv B(e,x;\bar{Y})$ is a desired one. □

By Lemmata 5.2 and 5.4 we get the

Lemma 5.5 *For each $\Phi \in POS$ there exists a formula $P^\Phi(x)$ in L such that*

$$EL + AC - NF \vdash \forall x\{P^\Phi(x) \leftrightarrow \Phi[\{x\}P^\Phi(x)/X]\}$$

Let $EL + AC - NF + \hat{ID}^i_n$ denote an extension of $EL + AC - NF$. Its language is obtained from L by adding a unary set constant I^Φ for each $\Phi \in POS^*(Y)$ (Y: a fixed set parameter) and its axioms are those of $EL + AC - NF$ in the expanded language plus the axiom $(FP)^\Phi_n$:

$$(FP)^\Phi_n \ \ \forall i < n\forall x[I^\Phi_i(x) \leftrightarrow \Phi(I^\Phi_{<i}, I^\Phi_i, x)]$$

where $I^\Phi_i(x) \equiv I^\Phi(j(i,x))$, $I^\Phi_{<i}(k,x) \equiv k < i \wedge I^\Phi_k(x)$ and $\Phi \equiv \Phi(Y,X,x)$.

Theorem 5.1 *$EL + AC - NF + \hat{ID}^i_n$ is a definitional extension of $EL + AC - NF$, i.e., the set constant I^Φ is definable in $EL + AC - NF$, and hence,*
via N. Goodman's theorem [12], $EL + AC - NF + \hat{ID}^i_n$ is a conservative extension of HA for each n.

Proof. Construct $P^\Phi_0, P^\Phi_1, \ldots, P^\Phi_{n-1}$ successively by Lemma 5.5. □

6 Classical fixed point theories

Let L_2 denote the second order language obtained from the language of the first order arithmetic by adding set variables $X, Y, \ldots$ Let $T \supseteq ACA_0$ denote a second order arithmetic containg ACA_0. Assume that T is $\Pi^1_1 - faithful$, i.e., any Π^1_1-consequence in T is true. Then, by [11], we have for a recursive theory T,

$$| T |=_{df} \sup\{\alpha : T \vdash I(\prec) \text{ for some recursive well ordering } \prec \text{ of type } \alpha\} < \omega^{CK}_1$$

where $I(\prec)$ denotes the Π^1_1-sentence $\forall X\, Prg[\prec, X] \rightarrow \forall x X(x)$. $Prg[\prec, X]$ denotes that X is progressive with respect to $\prec$ as in Section 3.
The *proof theoretic ordinal* $| T |$ of T is free from pathology, while the following alternative definition of the proof theoretic ordinal make sense relative to a vague natural well ordering $\prec$:

$$| T |_0 =_{df} \sup\{\alpha : T \vdash I(\prec, \Pi^{1-}_0) \text{ for some recursive well ordering } \prec \text{ of type } \alpha\}$$

where Π^{1-}_0 denotes the set of arithmetical formulae without set parameters and $I(\prec, \Pi^{1-}_0)$ the schema of transfinite induction of $\prec$ applied to a formula$\in \Pi^{1-}_0$.

Let $FP - ACA'_0$ and $FP - ACA'$ denote second order arithmetic in the language L_2 (without set constants P_A differing from [16]) which are obtained from ACA_0 and ACA, resp. by addind the following Σ^1_1 axiom:

$$(FP)\ \exists X\forall x(A[X,x] \leftrightarrow X(x))$$

for each X positive arithmetical formula $A[X,x]$ in L_2 ($A[X,x]$ contains no free variable except X and x.) Then G. Jäger and B. Primo [16] shows that

Theorem 6.1 (G. Jäger and B. Primo [16])

1. $| FP - ACA'_0 |= \varepsilon_0$
2. $| FP - ACA' |= \varepsilon_{\varepsilon_0}$
3. *$FP - ACA'_0$ and $\Sigma^1_1 - AC$ are proof theoretically equivalent each other.*

Here note that $| ACA_0 |= \varepsilon_0$, $| ACA |= \varepsilon_{\varepsilon_0}$ and $| \Sigma^1_1 - AC |= \varphi\varepsilon_0 0$. Also $FP - ACA'_0$ is proof theoretically stronger than ACA_0, e.g., by a truth definition for arithmetical formulae in Π^{1-}_0 $FP - ACA'_0 \vdash Con(ACA_0)$.

We observe that the above theorem follows from a result due to G. Kreisel[17] or [24], pp.176-177:

Theorem 6.2 *Let T be a recursive, Π^1_1-faithful second order arithmetic containg ACA_0.*

1. (G. Kreisel[17])

$$| T |= \sup\{\alpha : T \vdash I(\prec) \text{ for some } \Sigma^1_1 \text{ well ordering } \prec \text{ of type } \alpha\}$$

2. *Let $Tr_{\Sigma^1_1}$ denote the set of true Σ^1_1 sentences in L_2. Then*

$$| T |=| T + Tr_{\Sigma^1_1} |$$

Proof. Assume $T + A \vdash I(\prec)$ for a primitive recursive well ordering $\prec$ and an $A \in Tr_{\Sigma^1_1}$. Define a Σ^1_1 well ordering $\prec_A$ by

$$n \prec_A m \Leftrightarrow_{df} n \prec m \,\&\, A.$$

Then we have $T \vdash I(\prec_A)$. By the Kreisel's result, the order type of $\prec$ is equal to the order type of $\prec_A \leq | T |$. □

The theorem is applied to the nth fold iterated fixed point theory $FP_n - ACA'_0$. $FP_n - ACA'_0$ is obtained from ACA_0 by adding the Σ^1_1 axiom (FP_n):

$$(FP_n)\ \exists X_n, \ldots, X_1 \forall x \bigwedge_{1 \leq i \leq n} (x \in X_i \leftrightarrow A_i(X_i^+, X_1, \ldots, X_{i-1}, x))$$

for each X_i positive formula A_i in the language $L_2 + \{X_1, \ldots, X_i\}$.
$FP_n - ACA'$ is obtained from $FP_n - ACA'_0$ by adding the full induction schema $\Pi^1_\infty - IA$.

Corollary 6.1 *For any $n \in \omega$,*

1. $| FP_n - ACA'_0 |= \varepsilon_0$
2. $| FP_n - ACA' |= \varepsilon_{\varepsilon_0}$

Thus the theories $FP_n - ACA'_0$ is weak with respect to the proof theoretical ordinal $| T |$. But these are proof theoretically much stronger than ACA_0. In the following we compute the other proof theoretical ordinal $| FP_n - ACA'_0 |_0$, etc.
In what follows let $<$ denote a standard well ordering of type Γ_0 (the first strongly critical number). Ordinals$\leq \Gamma_0$ and their codes are identified and denoted by $\alpha, \beta, \ldots$

Definition 6.1 1. Let T be a first order theory containg PA. A first order theory $\hat{ID}(T)$ *(fixed point theory over T)* is defined as follows: The *language* $L_{\hat{ID}(T)}$ of $\hat{ID}(T)$ is obtained from the language L_T of T by adding the set constants $\{P_A : A[X^+, x] \in L_T(X),\ X \text{ positive } \}$.
Axioms $\hat{ID}(T) = T +$ induction schema for $L_{\hat{ID}(T)} + (FP)$

$$(FP)\ \forall x(x \in P_A \leftrightarrow A[P_A, x])$$

2. $\hat{ID}_0 = PA$ and $\hat{ID}_{n+1} = \hat{ID}(\hat{ID}_n)$.
3. $L^n = L_{\hat{ID}(T)}$ and $L^n_2 = L^n +$ second order variables $X, Y, \ldots$
4. *the norm of* $\hat{ID}_n$ $(n \neq 0)$ is defined to be the following ordinal with $| k |_A < \alpha \Leftrightarrow k \in I_A^{<\alpha}$:

$$\inf\{\alpha : \forall A[X, x] \in L^0(X) \forall k \in \omega[\hat{ID}_n \vdash k \in P_A \Rightarrow | k |_A < \alpha\}]$$

5. $FP_n - ACA = ACA$ for the language $L^n_2 + \hat{ID}_n$

Clearly $\hat{ID}_n$ and $FP_n - ACA'_0$ [$FP_n - ACA$ and $FP_n - ACA'$] have the same arithmetical provable formulae$\in L^0 = \Pi^{1-}_0$, resp. For a fixed $A[X^+, Y, x]$, we write P_n for P_{A_n} with $A_n = A[X^+, \sum_{i<n} P_i, x]$. Thus $\hat{ID}_n$ has extra constants $P_i\ (i < n)$ for each A.

Definition 6.2 Let Φ be a set of formulae.

1. $I(< \alpha, \Phi)$ denotes the schema of transfinite induction up to each $\beta < \alpha$ applied to a formula$\in \Phi$.

2. A first order theory $H(\Phi)^{<\alpha}$ is defined as follows: its *language*= L_0 + the language of $\Phi + \{H_A : A \in \Pi_0^{1-}(\Phi, X)\}$
   ($A \in \Pi_0^{1-}(\Phi, X) \Leftrightarrow_{df} A$ is a Π_0^1 formula relative to X formulae $\in \Phi$).
   $H(\Phi)^{<\alpha} = PA$ for the language of $H(\Phi)^{<\alpha} + (H)$:

$$(H)\ \forall x(x \in H_A^{\beta} \leftrightarrow A[H_A^{<\beta}, x])$$

   for each $\beta < \alpha$. $H_A^{<\beta} = \sum_{\gamma<\beta} H_A^{\gamma}$

Thus (H) says that $\{H_A^{\gamma} : \gamma \leq \beta\}$ forms the 'jump' hierarchy relative to formulae$\in \Phi$.

Theorem 6.3 *1. For each $B \in L^m$ $(m < n)$,*

$$\hat{ID}_n \vdash B \Leftrightarrow H(L^m)^{<\alpha_{n-m}} + \hat{ID}_m \vdash B \Leftrightarrow \hat{ID}_m + I(<\alpha_{n-m+1}, L^m) \vdash B$$

where $\alpha_1 = \varepsilon_0$, $\alpha_{n+1} = \varphi\alpha_n 0$ with the Veblen function $\varphi\alpha\beta$.

2. $|\hat{ID}_n|_0 = \alpha_{n+1}$

3. *For each $B \in L^m$ $(m < n)$,*

$$\begin{aligned} FP_n - ACA \vdash B \quad &\Leftrightarrow \quad H(L^m)^{<\beta_{n-m}} + \hat{ID}_m \vdash B \\ &\Leftrightarrow \quad \hat{ID}_m + I(<\beta_{n-m+1}, L^m) \vdash B \end{aligned}$$

where $\beta_1 = \varepsilon_{\varepsilon_0}$, $\beta_{n+1} = \varphi\beta_n 0$.

4. $|FP_n - ACA|_0 = \beta_{n+1}$

5. *the norm of $\hat{ID}_n = \alpha_n$ $(n \neq 0)$*

6. *the norm of $FP_n - ACA = \beta_n$*

This is proved by using usual techniques in [16] and [8].
Proof of 1 and 2. An infinitary system $\hat{ID}^{\infty}(L^n)$ ($\hat{ID}^{\infty}$ over the language L^n) is designed as the first order part of $FP - ACA^*$ in [16], in the language L^n, i.e., fixed points rules in $\hat{ID}^{\infty}(L^n)$ are only for P_n and constants $P_0, \dots, P_{n-1} \in L^n$ are treated as set parameters in $\hat{ID}^{\infty}(L^n)$. Thus $\hat{ID}^{\infty}(L^0)$ is the first order part of $FP - ACA^*$ in [16]. Put $B_n \equiv \bigwedge_{i<n} FP_i \supset B$ for $B \in L^{n+1}$ where FP_i denotes the axiom for the constant P_i. □

Lemma 6.1 *1. $\hat{ID}_{n+1} \vdash B \Rightarrow \hat{ID}^{\infty}(L^n) \vdash_1^{<\varepsilon_0} B_n$*

2. *$FP_{n+1} - ACA \vdash B \Rightarrow \hat{ID}^{\infty}(L^n) \vdash_1^{<\varepsilon_{\varepsilon_0}} B_n$*

For a proof we set the rank $rn(F) = 0$ if $F \in \mathcal{PN}_n$ with respect to P_n. The rest is the same in [16].

Lemma 6.2 *For an ε-number α,*

1. *$\hat{ID}^{\infty}(L^n) \vdash_1^{\alpha_0} B$ with $B \in \mathcal{PN}_n \Rightarrow \forall\beta < \alpha[(I_n^{<\alpha}) \vdash_{<\alpha}^{\alpha} \{\beta\}B\{\beta + \omega^{\alpha_0}\}]$ where $\{\beta\}B\{\beta'\}$* denotes the result of replacing each negative P_n by $I_n^{<\beta}$ and each positive P_n by $I_n^{<\beta'}$ *and* $(I_n^{<\alpha})$ is an infinitary system whose extra rules are, for each $\beta < \alpha$,

$$\frac{\Gamma, [\neg]A[I_n^{<\beta}, \sum_{i<n} P_i, s]}{\Gamma, [\neg]s \in I_n^{\beta}}$$

2. *$(I_n^{<\alpha}) \vdash_{<\alpha}^{\alpha} B \Rightarrow (I_n^{<\alpha}) \vdash_0^{<\varphi\alpha 0} B$*

3. *$\hat{ID}^{\infty}(L^n) \vdash_1^{<\alpha} B_n$ with $B \in L^n(P_n$* does not occur in $B)$
   $\Rightarrow \hat{ID}^{\infty}(L^{n-1}) \vdash_1^{<\varphi\alpha 0} B_{n-1}$

Lemma 6.3

$$H(X)^{<\omega^a} \vdash I(<\varphi a 0, X)$$

We give a sketch of a proof of this lemma below. From this lemma we see the

Lemma 6.4

$$H(L^m)^{<\alpha_{n-m}} \vdash I(<\alpha_{n-m+1}, L^m) \ \ (m \leq n, \alpha_0 = 0)$$

Thus we have shown the direction

$$\hat{ID}_m + I(<\alpha_{n-m+1}, L^m) \vdash B \ \Rightarrow \ H(L^m)^{<\alpha_{n-m}} + \hat{ID}_m \vdash B$$

Next consider the direction

$$\hat{ID}_n \vdash B \ \Rightarrow \ \hat{ID}_m + I(<\alpha_{n-m+1}, L^m) \vdash B$$

Assume $\hat{ID}_n \vdash B$ with $B \in L^m$, $m < n$. By Lemma 6.1 we have for $\alpha_1 = \varepsilon_0$ $\hat{ID}^\infty(L^{n-1}) \vdash_1^{<\alpha_1} B_n$. By Lemma 6.2 we successively have

$$\hat{ID}^\infty(L^m) \vdash_1^{<\alpha_{n-m}} B_m, \ (I_m^{<\alpha_{n-m}}) \vdash_{<\alpha_{n-m}}^{<\alpha_{n-m}} B_m \text{ and } (I_m^{<\alpha_{n-m}}) \vdash_0^{<\alpha_{n-m+1}} B_m.$$

By a patial truth definition we get $\hat{ID}_m + I(<\alpha_{n-m+1}, L^m) \vdash B$.

Finally consider the direction

$$H(L^m)^{<\alpha_{n-m}} + \hat{ID}_m \vdash B \Rightarrow \hat{ID}_n \vdash B$$

This follows from Lemma 6.5 below. We interprete $H(L^m)^{<\alpha_{n-m}} + \hat{ID}_m$ in $\hat{ID}_n$ as follows:

- leave L_m formulae unchanged.
- the 'jump' hierarchy H_A ($A \in \Pi_0^{1-}(L^m)$) up to$< \alpha_{n-m}$ is interpreted as P_A^+, P_A^- so that $P_A^+ = H_A$, $P_A^- = \neg H_A$ (simultaneously defined as fixed points over L^m). Then for each B in the language of $H(L^m)^{<\alpha_{n-m}} + \hat{ID}_m$ let B' denote the result of replacing the positive H_A by P_A^+ and negative H_A by P_A^-.

Lemma 6.5 $1)_m$ $H(L^m)^{<\alpha_{n-m}} + \hat{ID}_m \vdash B \Rightarrow \hat{ID}_n \vdash B'$, *i.e.,*
$\hat{ID}_n \vdash \forall x(x \in H_A^\beta \leftrightarrow \neg(x \notin H_A^\beta))$ *for each* $\beta < \alpha_{n-m}$.

$2)_m$ $\hat{ID}_n \vdash I(<\alpha_{n-m+1}, L^m)$ $(m \leq n)$

Proof by simultaneous induction on $n-m$. We have $2)_n$ and $2)_{m+1} \Rightarrow 1)_m$. It remains to show $1)_m \Rightarrow 2)_m$. By Lemma 6.4 and $I(<\alpha_{n-m+1}, L^m) \in L_m$ we get $2)_m$. □

Thus we have proven Theorem 6.3.1 and 2.

Finally consider the norm of $\hat{ID}_n$. The upper bound α_n for the norm of $\hat{ID}_n$ is obtained from Lemmata 6.1 and 6.2.
To obtain the lower bound, define a fixed point $W = W_0$ by

$$\forall\beta(\beta \in W \leftrightarrow \forall\gamma < \beta(\gamma \in W))$$

By Lemma $6.5.2)_1$, we have $\hat{ID}_n \vdash I(<\alpha_n, L^1)$. Hence by $W \in L^1$ and
$\hat{ID}_n \vdash \forall\beta(\forall\gamma < \beta(\gamma \in W) \to \beta \in W)$, we get $\hat{ID}_n \vdash \beta \in W$ for each $\beta < \alpha_n$.

Proof of Lemma6.3. Put $\lambda = \omega^a$ and

$$I_X^{<\beta}(\gamma) \Leftrightarrow_{df} \forall Y \in \bigcup_{\delta<\beta} Rec(H_\delta^X) I(\gamma, Y)$$

where

1. H_δ^X denotes the δ^{th} jump of the set X
2. $Rec(H_\delta^X)$ denotes the set of sets recursive in H_δ^X.
3. $I(\gamma, Y)$ denotes the transfinite induction up to γ applied to Y.

Also for each $\alpha < \lambda$,

$$A_\alpha^X(\gamma) \Leftrightarrow_{df} \gamma > 0 \to \forall\beta\forall\delta > 0[I_X^{\omega^\gamma(\delta+1)}(\beta) \,\&\, \omega^\gamma(\delta+1) \leq \alpha \to I_X^{<\omega^\gamma\delta}(\varphi\gamma\beta)]$$

Then we can prove the following lemma as in [8]:

Lemma 6.6 *For each $\alpha < \lambda$,*

$$H(X)^{<\lambda} \vdash Prg[A^X_\alpha]$$

Lemma 6.7

$$H(X)^{<\lambda} \vdash I(<\lambda)$$

where $I(<\lambda)$ denotes the schema of transfinite induction up to each ordinal$< \lambda$ and applied to any formula in the language of $H(X)^{<\lambda}$.

Proof. For $\alpha < \lambda$ let $S\alpha$ denote a finite set of ordinals$\leq \alpha$ inductively generated as follows:

1. $\alpha \in S\alpha$
2. If $\beta = \beta_1 + \cdots + \beta_n \in S\alpha$, $\beta_1 > \cdots > \beta_n$ & $\beta_1, \dots, \beta_n$ are additive principal, then $\beta_1, \dots, \beta_n \in S\alpha$.
3. If $\varphi\gamma\delta \in S\alpha$, then $\gamma, \delta \in S\alpha$.

We show inductively that $\forall\beta \in S\alpha\, H(X)^{<\lambda} \vdash I(\beta)$.
Assume that $\gamma > 0$ & $\varphi\gamma\delta \in S\alpha$, $I(\gamma)$ and $I(\delta)$. For a given formula U we have to show $I(\varphi\gamma\delta, U)$. Since $\lambda = \omega^a$ is additive principal, $\omega^\gamma \cdot 2 < \omega^{\varphi\gamma\delta} \cdot 2 = \varphi\gamma\delta \cdot 2 \leq \alpha \cdot 2 < \lambda$. Also $H(U)^{<\lambda} = H(X)^{<\lambda}$ since $U \in \bigcup_{\delta<\lambda} Rec(H^X_\delta)$ and λ is additive principal. Thus by Lemma 6.6 we have $Prg[A^U_{\alpha\cdot 2}]$. By $I(\gamma)$, we have $A^U_{\alpha\cdot 2}(\gamma)$ and hence $\forall\beta[I_U^{<\omega^\gamma 2}(\beta) \to I_U^{<\omega^\gamma}(\varphi\gamma\beta)]$. By $I(\delta)$ we have $I_U^{<\omega^\gamma 2}(\delta)$. Thus $I_U^{<\omega^\gamma}(\varphi\gamma\delta)$ and $I(\varphi\gamma\delta, U)$. □
Now Lemma 6.3 follows from Lemmata 6.6 and 6.7.

7 Iterated reflection formulae and rules of transfinite induction

In this section we give an equivalence between transfinite induction rule and iterated reflection schema over the fragment $I\Sigma_n$ of PA.
In this section $<$ denotes a standard ε_0 well ordering.

Definition 7.1 *1.* For an additive principal number $\alpha \geq \omega$ and a set Φ of formulae, $TIR[\alpha, \Phi]$ denotes the transfinite induction rule up to α and applied to a formula $A \in \Phi$: Put $Prg[A] \Leftrightarrow_{df} \forall x(\forall y < x A(y) \supset A(x))$. Then for each $A \in \Phi$

$$\frac{Prg[A]}{\forall x < \alpha A(x)}$$

is an instance of the rule $TIR[\alpha, \Phi]$.

2. For a theory T containg the fragment $I\Sigma_1$ let $T + TIR[\alpha, \Phi]$ denote the theory obtained from T by adding the rule $TIR[\alpha, \Phi]$. Also $T + TIR^{(m)}[\alpha, \Phi]$ $(m \in \omega)$ denotes a formal system$\subseteq T + TIR[\alpha, \Phi]$ in which the rule $TIR[\alpha, \Phi]$ can be applied nestedly at most m times.
For example 0) $T + TIR^{(0)}[\alpha, \Phi] = T$ and 1) in $T + TIR^{(1)}[\alpha, \Phi]$ the rule $TIR[\alpha, \Phi]$ can be applied only when $T \vdash Prg[A]$ $(A \in \Phi)$, etc.

3. For a theory $T \supseteq I\Sigma_1$ let $C^T_n(\alpha)$ denote the iterated reflection formula defined in U. Schmerl [19]. Thus in $I\Sigma_1$ we have

(a) $C^T_n(0) \leftrightarrow RFN_{\Pi_{n+1}}(T)$
(b) $C^T_n(\alpha + 1) \leftrightarrow RFN_{\Pi_{n+1}}(T + C^T_n(\alpha))$
(c) $C^T_n(\lambda) \leftrightarrow \forall\alpha < \lambda C^T_n(\alpha)$ for a limit λ.

4. $\begin{pmatrix} n \\ \alpha \end{pmatrix}_T =_{df} T + \{C^T_n(\beta) : \beta < \alpha\}$ as in [19].

Proposition 7.1 *Over $I\Sigma_1$,*

$$TIR[\omega^{2+\alpha}, \Sigma_n] = TIR[\omega^{1+\alpha}, \Pi_{n+1}]$$

Proof. This is contained in the proof of Theorem 4.1. e) in [23] □

A formula $A(x)$ is called *reflexively progressive* (in x) with respect to a theory T if

$$T \vdash \forall x[\forall y < x Pr_T(\text{"}A(\dot{y})\text{"}) \supset A(x)]$$

with a canonical provability predicate Pr_T for T and the gödel number "E" of an expression E.

Proposition 7.2 *(cf. [19],p.337)*

$$T \vdash A(x) \Leftrightarrow A(x) \textit{ is reflexively progressive with respect to } T$$

Remark. The proof of the direction [$\Leftarrow$] in [19] uses Löb's theorem and the facts:

1. $T \vdash y < z \supset Pr_T(" \dot{y} < \dot{z}")$
2. $T \vdash <$ is transitive

Thus any Σ_1 binary relation $<$ Proposition 7.2 holds if $<$ is demonstrably transitive in T. In other words, reflexive progressiveness is nothing to well foundedness although the name remind us the latter.

Lemma 7.1 *For $A \in \Pi_{n+1}$ and $T \supseteq I\Sigma_1$,*

1. *$B(\alpha) \equiv C_n^T(\alpha) \supset A(\alpha)$ is reflexively progressive with respect to T if $T \vdash C_n^T(0) \supset Prg[A]$.*
2. *$T \vdash Prg[A] \Rightarrow T + C_n^T(0) \vdash Prg[\forall x < \omega(1+\alpha)A(x)]$.*
3. *$T \vdash Prg[A] \Rightarrow T \vdash C_n^T(\alpha) \supset \forall x < \omega(1+\alpha)A(x)$.*

Proof.

1. Assume $T \vdash C_n^T(0) \supset Prg[A]$. We can assume that $\alpha \neq 0$ since $T \vdash C_n^T(0) \supset A(0)$. Then we have by $A \in \Pi_{n+1}$,
$$T \vdash \forall \beta < \alpha Pr_T(" C_n^T(\dot{\beta}) \supset A(\dot{\beta})") \,\&\, C_n^T(\alpha) \supset \forall \beta < \alpha A(\beta)$$
By our assumption $T \vdash C_n^T(\alpha) \supset Prg[A]$.
2. Assume $T \vdash Prg[A]$. Consider the case $\alpha = 0$. Then we have to show $T + C_n^T(0) \vdash \forall x < \omega A(x)$. This follows from $T \vdash \forall n < \omega Pr_T(" A(\dot{n})")$ or better $T \vdash \forall n < \omega Pr_T("\forall x < \dot{n} A(x)")$. Other cases are similar.

□

Lemma 7.2 *Assume $T \supseteq I\Sigma_n$ and A is a Π_{n+1}-sentence. Then*

$$T \vdash A \Rightarrow T + TIR^{(1)}[\omega^{1+\alpha}, \Pi_{n+1}] \vdash C_n^{I\Sigma_n + A}(\underline{\omega^\alpha})$$

with

$$\underline{\omega^\alpha} =_{df} \begin{cases} \omega^\alpha & \alpha \neq 0 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Let $B(\omega\beta + p)$ denote the Π_{n+1}-formula:

$$\beta < \omega^\alpha \,\&\, p < \omega \,\&$$
$$\forall \Gamma \subseteq \Pi_{n+1}\{Prov_{I\Sigma_n}(p, "\neg A, \neg C_n^{I\Sigma_n}(\dot{\beta} - 1), \Gamma") \supset Tr_{\Pi_{n+1}}(" \bigvee \Gamma")\}$$

where

1. Π_{n+1} =the set of gödel numbers of Π_{n+1}-formulae
2. $Prov_{I\Sigma_n}(p, "\Gamma")$ is a proof predicate for $I\Sigma_n$ which says that p is a proof of a sequent Γ in $I\Sigma_n$. Here $I\Sigma_n$ is formulated in a Tait's calculus.
3. $Tr_{\Pi_{n+1}}$ denotes a partial truth definition for Π_{n+1}-fomulae.
4. $\beta - 1 =_{df}$if $\beta = n < \omega$ then $n - 1$ else β, and $C_n^{I\Sigma_n}(-1)$ denotes a true formula, e.g., $0 = 0$.

We assume that when $\Gamma \subseteq \Pi_{n+1}$ and $Prov_{I\Sigma_n}(p, "\neg A, \neg C_n^{I\Sigma_n}(\dot{\beta} - 1), \Gamma")$, every sequent in the proof p is of the form $\neg A, \neg C_n^{I\Sigma_n}(\beta - 1), \Delta$ for some $\Delta \subseteq \Pi_{n+1}$. This follows from a partial cut elimination which is available in $I\Sigma_1 \subseteq T$.

We show that $T \vdash Prg[B]$. Argue in T. We have A and $\forall \gamma < \beta \forall p < \omega B(\omega\gamma + p)$. Hence $C_n^{I\Sigma_n}(\beta - 1)$. By induction on $p < \omega$ we get $\bigvee \Gamma$. If a Σ_{n+1}-formula$\in \{\neg A, \neg C_n^{I\Sigma_n}(\beta - 1)\}$ is analysed by an inference rule ($\exists$), then use the fact: A and $C_n^{I\Sigma_n}(\beta - 1)$ are true. □

Theorem 7.1 *For each $\alpha \geq 0$ and $0 < n, m < \omega$,*

$$I\Sigma_n + TIR^{(m)}[\omega^{1+\alpha}, \Pi_{n+1}] = I\Sigma_n + C_n^{I\Sigma_n}(\underline{\omega^\alpha \cdot m})$$

with

$$\underline{\omega^\alpha \cdot m} =_{df} \begin{cases} \omega^\alpha \cdot m & \alpha \neq 0 \\ m-1 & \textit{otherwise} \end{cases}$$

Proof. $[\subseteq]$ By induction on $m \geq 0$, we show, for $A \in \Pi_{n+1}$

$$I\Sigma_n + C_n^{I\Sigma_n}(\underline{\omega^\alpha \cdot m}) \vdash Prg[A] \Rightarrow I\Sigma_n + C_n^{I\Sigma_n}(\underline{\omega^\alpha \cdot (m+1)}) \vdash \forall x < \omega^{1+\alpha} A(x)$$

where $I\Sigma_n + C_n^{I\Sigma_n}(\underline{\omega^\alpha \cdot 0}) = I\Sigma_n$. Put $T = I\Sigma_n + C_n^{I\Sigma_n}(\underline{\omega^\alpha \cdot m})$. By Lemma 7.1 $T \vdash C_n^T(\underline{\omega^\alpha}) \supset \forall x < \omega^{1+\alpha} A(x)$. Also $T + C_n^T(\underline{\omega^\alpha}) = I\Sigma_n + C_n^{I\Sigma_n}(\underline{\omega^\alpha \cdot (m+1)})$.
$[\supseteq]$ This follows from Lemma 7.2. □

In what follows we concentrate on the case $n = 1$. For a limit ordinal $\lambda < \varepsilon_0$, $\{\lambda[x]\}_{x \in \omega}$ denotes the fundamental sequence given in the Definition 3.7 in [23], i.e., $\omega^{\alpha+1}[x] = \omega^\alpha \cdot (x+1)$.

Definition 7.2 *Fast growing functions F_α.*

1. F_α.
   (a) $F_0(x) = 2x + 2$
   (b) $F_{\alpha+1}(x) = F_\alpha^{(x)}(2)$
   (c) $F_\lambda(0) = 2$
   (d) $F_\lambda(x) = F_{\lambda[x]}(x)$ for a limit λ and $x \neq 0$
2. $F_\alpha(x) \downarrow$ denotes a Σ_1 formula saying '$F_\alpha(x)$ is defined'.
3. $F_\alpha \downarrow \Leftrightarrow_{df} \forall x \in \omega (F_\alpha(x) \downarrow)$: a Π_2 formula

R. Sommer [23] shows that the graph $\{(\alpha, x, y) : F_\alpha(x) = y\}$ is Δ_0 definable.

Definition 7.3 *$Tot(T)$, $PR(\mathcal{F})$ and $ER(\mathcal{F})$.*

1. For a theory $T \supseteq I\Sigma_1$, $Tot(T)$ denotes the set of provably total recursive functions in T.
2. For a set $\mathcal{F}$ of functions on ω, $PR(\mathcal{F})$ $[ER(\mathcal{F})]$ denotes the primitive [elementary] recursive closure of $\mathcal{F}$, resp.

Lemma 7.3 *1. Each $f \in Tot(I\Sigma_1 + F_\alpha \downarrow)$ is majorized by an $F_{\alpha+n}$ for some $n < \omega$. Thus $Tot(I\Sigma_1 + F_\alpha \downarrow) \subseteq PR(F_\alpha)$.*

2. $I\Sigma_1 \vdash F_\alpha \downarrow \rightarrow F_{\alpha+1} \downarrow$. Thus $Tot(I\Sigma_1 + F_\alpha \downarrow) = PR(F_\alpha)$.

3. $I\Sigma_1 \vdash F_{\alpha+\omega} \downarrow \leftrightarrow RFN_{\Pi_2}(I\Sigma_1 + F_\alpha \downarrow) = C_1^{I\Sigma_1 + F_\alpha\downarrow}(0)$

Proof. 2. It suffices to show $I\Sigma_1 + F_\alpha \downarrow \vdash \forall y \forall x (F_\alpha^{(x)}(y) \downarrow)$. Fix y as a parameter and use $I\Sigma_1$ to show $\forall x (F_\alpha^{(x)}(y) \downarrow)$ by induction on x.
3. $[\rightarrow]$ by a formalization of a proof of Lemma 7.3.1 in $I\Sigma_1$. $[\leftarrow]$ follows from 2. □

Lemma 7.4

$$I\Sigma_1 \vdash C_1^{I\Sigma_1}(\alpha) \leftrightarrow F_{\omega(1+\alpha)} \downarrow$$

Proof. $[\rightarrow]$ By the Lemma 7.1.3, it suffices to show $I\Sigma_1 \vdash Prg[A]$ with $A(x) \Leftrightarrow_{df} F_x \downarrow \in \Pi_2$. This follows from the Lemma 7.3.2.
$[\leftarrow]$ Put $B(\alpha) \Leftrightarrow_{df} F_{\omega(1+\alpha)} \downarrow \rightarrow C_1^{I\Sigma_1}(\alpha)$. We show this formula $B(\alpha)$ is reflexively progressive with respect to $I\Sigma_1$. Argue in $I\Sigma_1$ and assume that

$$\forall \beta < \alpha Pr_{I\Sigma_1}(\text{"}B(\dot{\beta})\text{"}) \,\&\, F_{\omega(1+\alpha)} \downarrow .$$

Case 0. $\alpha = 0$: By the Lemma 7.3.3, $F_\omega \downarrow \rightarrow C_1^{I\Sigma_1}(0)$
Case 1. $\alpha \neq 0$: Assume $\beta < \alpha \,\&\, Pr_{I\Sigma_1}(\text{"}C_1^{I\Sigma_1}(\dot{\beta}) \rightarrow A\text{"})$ for a $A \in \Pi_2$. By a cut, $Pr_{I\Sigma_1}(\text{"}F_{\omega(1+\dot{\beta})} \downarrow \rightarrow A\text{"})$. By $\omega(1+\beta) + \omega \leq \omega(1+\alpha)$, we see $F_{\omega(1+\beta)+\omega} \downarrow$ from $F_{\omega(1+\alpha)} \downarrow$. Again by the Lemma 7.3.3, we have $RFN_{\Pi_2}(I\Sigma_1 + F_{\omega(1+\beta)} \downarrow)$. Thus $Tr_{\Pi_2}(\text{"}A\text{"})$. □

Observe that $\omega^{1+\alpha} \cdot m = \omega(1 + \underline{\omega^\alpha \cdot m})$. Therefore from these lemmata and the Theorem 7.1 we see the

Theorem 7.2 *For each $\alpha \geq 0$ and $0 < m < \omega$,*

$$T_\alpha^{(m)} =_{df} I\Sigma_1 + TIR^{(m)}[\omega^{1+\alpha}, \Pi_2] = I\Sigma_1 + C_1^{I\Sigma_1}(\underline{\omega^\alpha \cdot m}) = I\Sigma_1 + F_{\omega^{1+\alpha}\cdot m} \downarrow$$

and

$$Tot(T_\alpha^{(m)}) = PR(F_{\omega^{1+\alpha}\cdot m})$$

Corollary 7.1 *For $0 \leq k, m < \omega$ with $m \neq 0$,*

$$T_k^{(m)} =_{df} I\Sigma_1 + TIR^{(m)}[\omega^{1+k}, \Pi_2] = I\Sigma_1 + C_1^{I\Sigma_1}(\underline{\omega^k \cdot m}) = I\Sigma_1 + F_{\omega^{1+k}\cdot m} \downarrow$$

and

$$Tot(T_k^{(m)}) = PR(F_{\omega^{1+k}\cdot m})$$

8 Derivation lengths of finite rewrite rules reducing under lexicographic path orders

In this section we discuss a relationship between the derivation lengths of finite rewrite rules reducing under lexicographic path orders and the provably total recursive functions in theories $T_k^{(m)}$ defined in Corollary 7.1. In Weiermann [26] and Buchholz [2] it is shown that

Theorem 8.1 (Weiermann [26] and Buchholz [2])
The derivation lengths of finite rewrite rules reducing under a lexicographic path order are bounded by a multiply recursive function $F_{\omega^{1+k}\cdot m}$ $(k, m \in \omega)$.

First we introduce a variant of a slow growing function $G_n\alpha$ in [1].

Definition 8.1 *1. Od, P and $S\alpha \in \{0,1\}$.*

- *(a)* $P \subset Od$.
- *(b)* $0 \in Od$, $S0 = 0$. $[S\alpha = 0 \Leftrightarrow \alpha < \Omega.]$
- *(c)* $\alpha_1, \dots, \alpha_n \in P \,\&\, \alpha_1 \geq \cdots \geq \alpha_n \,(n \geq 2) \Rightarrow \alpha_1 + \cdots + \alpha_n \in Od$.
  [Here $\alpha \leq \beta \Leftrightarrow_{df} \alpha < \beta$ or $\alpha = \beta$.]
  $S(\alpha_1 + \cdots + \alpha_n) = \max\{S\alpha_i : 1 \leq i \leq n\} = S\alpha_1$.
- *(d)* $\alpha \in Od|\Omega =_{df} \{\alpha \in Od : \alpha < \Omega\} \Rightarrow \omega^\alpha \in P$. $S\omega^\alpha = S\alpha = 0$.
- *(e)* $\alpha \in Od \Rightarrow d\alpha \in P$. $Sd\alpha = 0$.
- *(f)* $0 < n < \omega \,\&\, \xi \in P|\Omega = \{\xi \in P : \xi < \Omega\} \Rightarrow \Omega^n \cdot \xi \in P$. $S\Omega^n \cdot \xi = 1$.

2. $K\alpha \subset P|\Omega$

- *(a)* $K0 = \emptyset$
- *(b)* $K(\alpha_1 + \cdots + \alpha_n) = \bigcup\{K\alpha_i : 1 \leq i \leq n\}$
- *(c)* $K\omega^\alpha = K\alpha$
- *(d)* $Kd\alpha = \{d\alpha\}$
- *(e)* $K(\Omega^n \cdot \xi) = K\xi$

3. $\alpha < \beta$

- *(a)* $\beta \neq 0 \Rightarrow 0 < \beta$
- *(b)* $\alpha_1 + \cdots + \alpha_n < \beta_1 + \cdots + \beta_m \,(\alpha_i, \beta_j \in P \,\&\, n + m > 2) \Leftrightarrow$
  - *i.* $n < m \,\forall i < n(\alpha_i = \beta_i)$ or
  - *ii.* $\exists l \leq \min\{n, m\}[\alpha_l < \beta_l \,\&\, \forall i < l(\alpha_i = \beta_i)]$
- *(c)* $\alpha \in P|\Omega \Rightarrow \alpha < \Omega^m \cdot \zeta$
- *(d)* $\alpha < d\beta \Rightarrow \omega^\alpha < d\beta$, and $d\alpha \leq \beta \Rightarrow d\alpha < \omega^\beta$
- *(e)* $\alpha < \beta (< \Omega) \Rightarrow \omega^\alpha < \omega^\beta$
- *(f)* $d\alpha < d\beta \Leftrightarrow$
  - *i.* $\alpha < \beta \,\&\, K\alpha < d\beta$ or

ii. $d\alpha \leq K\beta$
$[X < \beta \Leftrightarrow_{df} \forall\alpha \in X(\alpha < \beta)$ and $\alpha \leq Y \Leftrightarrow_{df} \exists\beta \in Y(\alpha \leq \beta)$]

(g) $\Omega^n \cdot \xi < \Omega^m \cdot \zeta \Leftrightarrow$
 i. $n < m$ or
 ii. $n = m \,\&\, \xi < \zeta$

4. *Conventions*
 (a) $1 = \omega^0$, $n = 1 + \cdots + 1$ for $n < \omega$.
 (b) $\Omega^n \cdot 0 = 0$, $\Omega^0 \cdot \xi = \xi$, $\Omega^n = \Omega^\alpha \cdot 1$ and $\Omega = \Omega^1$.
 (c) $\Omega^m \cdot (\xi_1 + \cdots + \xi_n) = \Omega^m \cdot \xi_1 + \cdots + \Omega^m \cdot \xi_n$
 for $\Omega > \xi_1 \geq \cdots \geq \xi_n$, $\xi_1, \ldots, \xi_n \in P$.
 (d) $\alpha \in P_\Omega \Leftrightarrow_{df} [\alpha < \Omega \,\&\, \alpha \in P]$ or $[\alpha = \Omega^n \cdot \xi \geq \Omega$ for some $n, \xi]$
 (e) $\alpha \# \beta$ denotes the *natural sum*.

Definition 8.2 *Normal Forms*

1. We write $\alpha =_{NF_0} \alpha_1 + \cdots + \alpha_n$
 if $n \geq 1$, $\alpha = \alpha_1 + \cdots + \alpha_n$, $\alpha_1 \geq \cdots \geq \alpha_n \,\&\, \forall i \leq n(\alpha_i \in P)$

2. For each $\alpha \in Od$ with $\alpha \neq 0$, $\exists! n < \omega \, \exists!(\alpha_0, \ldots, \alpha_n) \, \exists!(\xi_0, \ldots, \xi_n)$ such that

$$\alpha = \Omega^{\alpha_n} \cdot \xi_n + \cdots + \Omega^{\alpha_0} \cdot \xi_0 \,\&\, 0 = \alpha_0 < \cdots < \alpha_n < \omega$$
$$0 < \xi_1, \ldots, \xi_n < \Omega \,\&\, 0 \leq \xi_0 < \Omega$$

 In this case we write

$$\alpha =_{\Omega-NF} \Omega^{\alpha_n} \cdot \xi_n + \cdots + \Omega^{\alpha_0} \cdot \xi_0 =_{\Omega-NF} \sum_{i=0}^{n} \Omega^{\alpha_i} \cdot \xi_i$$

3. For each $\alpha \in Od$ with $\alpha \neq 0$,
 $\exists! n < \omega \, \exists! m < \omega \, \exists!(\alpha_1, \ldots, \alpha_n) \, \exists!(\xi_1, \ldots, \xi_n) \exists!(\beta_1, \ldots, \beta_m)$ such that

$$\alpha = \sum_{i=1}^{n} \Omega^{\alpha_i} \cdot \xi_i + \sum_{i=1}^{m} \beta_i \,\&\, 0 < \alpha_n < \cdots < \alpha_1 < \omega \,\&0 < \xi_1, \ldots, \xi_n < \Omega$$
$$\beta_m \leq \cdots \leq \beta_1 < \Omega \,\&\, \forall i \leq m(\beta_i \in P) \,\&\, n + m > 0 \, (n, m \geq 0)$$

 In this case we write

$$\alpha =_{NF_1} \sum_{i=1}^{n} \Omega^{\alpha_i} \cdot \xi_i + \sum_{i=1}^{m} \beta_i =_{NF_1} \sum_{i=1}^{n} \gamma_i + \sum_{i=1}^{m} \beta_i \text{ with } \gamma_i = \Omega^{\alpha_i} \cdot \xi_i$$

Definition 8.3 The *norm* $N\alpha$ of $\alpha \in Od$

1. $N0 = 0$
2. $N\alpha = \max\{n, N\alpha_i : 1 \leq i \leq n\}$ for $\alpha =_{NF_0} \alpha_1 + \cdots + \alpha_n < \Omega$
3. $N\omega^\alpha = N\alpha + 1$
4. $Nd\alpha = N\alpha + 1$
5. $N\alpha = \max(\{k - 1\} \cup \{N\xi_i : i \leq n\})$ for $\alpha =_{\Omega-NF} \Omega^{\alpha_n} \cdot \xi_n + \cdots + \Omega^{\alpha_0} \cdot \xi_0$ with $0 < k = \alpha_n < \omega$.

Definition 8.4 *(cf. [2], [26])* $\alpha <_k \beta$

1. $\beta \neq 0 \Rightarrow 0 <_k \beta$ *[zero]*
2. $\beta =_{NF_1} \beta_1 + \cdots + \beta_m$ $(\beta_i \in P_\Omega, m \geq 2)$:
 $\exists(\alpha_1, \ldots, \alpha_m)[\alpha = \alpha_1 \# \cdots \# \alpha_m \,\&\, \forall i \leq m(\alpha_i \leq_k \beta_i) \,\&\, \exists i \leq m(\alpha_i <_k \beta_i)$
 $\Rightarrow \alpha <_k \beta$ *[multiset]*
 [Here α_i may be 0 and/or $\notin P_\Omega$. $\alpha \leq_k \beta \Leftrightarrow_{df} \alpha <_k \beta$ or $\alpha = \beta$.]
3. $\beta \in P_\Omega \,\&\, \alpha =_{NF_1} \alpha_1 + \cdots + \alpha_n$ $(\alpha_i \in P_\Omega, n \geq 2)$:

(a) $S\beta = 0$: $\forall i \leq n(\alpha_i <_k \beta)$ & $N\alpha \leq N\beta + k \Rightarrow \alpha <_k \beta$ [*inaccessibility*]

(b) $S\beta = 1$: $\forall i \leq n(\alpha_i <_k \beta) \Rightarrow \alpha <_k \beta$ [*additive principal*]

4. $\alpha, \beta \in P_\Omega$ & $0 = S\alpha < S\beta = 1 \Rightarrow \alpha <_k \beta$ [*Stufe*]

5. $\alpha, \beta \in P_\Omega$ & $S\alpha = S\beta = 1$ & $\alpha =_{\Omega-NF} \Omega^n \cdot \xi$ & $\beta =_{\Omega-NF} \Omega^m \cdot \zeta$:

   (a) $n < m$ or

   (b) $\alpha_1 = \beta_1$ & $\xi <_k \zeta$

   $\Rightarrow \alpha <_k \beta$ [*lexicographical*]

6. $\alpha <_k \beta < \Omega \Rightarrow \omega^\alpha <_k \omega^\beta$ [*monotonicity*]

7. $d\alpha \leq_k \beta < \Omega \Rightarrow d\alpha <_k \omega^\beta$ [*subterm*]

8. $\alpha <_k d\beta$ & $N\omega^\alpha \leq Nd\beta + k \Rightarrow \omega^\alpha <_k d\beta$ [*inaccessibility*]

9. (a) $\alpha <_k \beta$ & $K\alpha <_k d\beta$ & $N\alpha \leq N\beta + k$ [*inaccessibility*] or

   (b) $d\alpha \leq_k K\beta$ [*subterm*]

   $\Rightarrow d\alpha <_k d\beta$

Lemma 8.1 *1. $N\alpha$ is a norm, i.e., the set $\{\beta \in Od : \beta < \alpha \,\&\, N\beta \leq n\}$ is finite for each $\alpha \in Od$ and $n \in \omega$.*

2. The set $\{\beta \in Od : \beta <_k \alpha\}$ is finite for each $\alpha < \Omega$ and $k \in \omega$.

Definition 8.5 *$G_n\alpha$ for $\alpha \in Od|\Omega$*

$$G_n\alpha =_{df} \max\{k \in \omega : \exists(\alpha_0, \ldots, \alpha_k)[\alpha_k <_n \cdots <_n \alpha_0 = \alpha]\}$$

First we show that the function $G_n\alpha$ is provably total in the fragments $T_k^{(m)}$ of $I\Sigma_2$.

Definition 8.6 *(cf.[2])*

1. $D_k =_{df} \{(\alpha_0, \ldots, \alpha_l) \subset Od|\Omega : \forall j \leq l \forall \alpha <_k \alpha_j(\alpha \in (\alpha_0, \ldots, \alpha_{j-1}))\}$

2. $W_k =_{df} \{\alpha \in Od|\Omega : \exists d \in D_k(\alpha \in d)\}$

3. $A_k(X, \alpha) \Leftrightarrow_{df} \alpha < \Omega \,\&\, \forall \beta <_k \alpha(\beta \in X)$ for a unary X

4. $A_k(X) =_{df} \{\alpha \in Od|\Omega : A_k(X, \alpha)\}$

Note that $D_k, W_k, M_k \in \Sigma_1$ and $A_k(X, \alpha) \in \Sigma_0(X^+)$. The following lemmata are seen as in [1].

Lemma 8.2 (W_k.1) $I\Sigma_1 \vdash A_k(W_k) = W_k$
(W_k.2) *For each $F \in \Sigma_1 \cup \Pi_1$,*

$$I\Sigma_1 \vdash A_k(F) \subseteq F \rightarrow W_k \subseteq F$$

and

$$I\Sigma_1 \vdash \forall \alpha \in W_k(\forall \beta <_k \alpha F(\beta) \rightarrow F(\alpha)) \rightarrow W_k \subseteq F.$$

Lemma 8.3 ($I\Sigma_1$) $\alpha, \beta \in W_k \leftrightarrow \alpha \# \beta \in W_k$

Lemma 8.4 ($I\Sigma_1$) $\beta =_{NF_0} \beta_1 + \cdots + \beta_n \,\&\, \forall i \leq n(\beta_i \in W_k) \rightarrow \beta \in W_k$

Lemma 8.5 ($I\Sigma_1$) $\alpha \in W_k \rightarrow \omega^\alpha \in W_k$

Lemma 8.6 *For $\{\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n\} \subset Od \mid \Omega$,*

$$\sum_{i=0}^{n} \Omega^i \cdot \alpha_i <_k \sum_{i=0}^{n} \Omega^i \cdot \beta_i \quad \Leftrightarrow \quad (\alpha_n, \ldots, \alpha_0) <_k^{lex} (\beta_n, \ldots, \beta_0)$$

$$\Leftrightarrow_{df} \quad \exists l \leq n[\alpha_l <_k \beta_l \,\&\, \forall i(l < i \leq n \rightarrow \alpha_i = \beta_i)]$$

Lemma 8.7 *For each k, m with $0 \leq k, m < \omega \,\&\, m \neq 0$,*

$$T_k^{(m)} \vdash \forall \alpha_0, \ldots, \alpha_{k+1} \in W_n\{d(\Omega^{2+k} \cdot (m-1) + \sum_{i=0}^{1+k} \Omega^i \cdot \alpha_i) \in W_n\}$$

Proof by induction on $m > 0$. Argue in $T_k^{(m-1)}$. Assume $d_0, \ldots, d_{k+1} \in D_n$, $d_i = (\beta_0^i, \ldots, \beta_{l_i-1}^i)$ with $l_i = lh(d_i)$. Show the Σ_1 formula

$$B(j_0, \ldots, j_{k+1}) \Leftrightarrow_{df} d(\Omega^{2+k} \cdot (m-1) + \sum_{i=0}^{1+k} \Omega^i \cdot \beta_{j_i}^i) \in W_n$$

is progressive with respect to the lexicographic order for $j_i < l_i \, (i \leq k+1)$. Then the rule $TIR[\omega^{1+k}, \Pi_2] = TIR[\omega^{2+k}, \Sigma_1]$ implies the assertion.
For a proof of the progressiveness use a subsidiary induction on $\ell\alpha$ for $\alpha <_n d(\Omega^{2+k} \cdot (m-1) + \sum_{i=0}^{1+k} \Omega^i \cdot \beta_{j_i}^i)$ and the Lemmata 8.4,8.5 and 8.6. □

Lemma 8.8 *For each $l < \omega$,*

$$T_k^{(m)} \vdash \forall \alpha \in W_n(d(\Omega^{2+k} \cdot m + \Omega l + \alpha) \in W_n)$$

Proof by metainduction on $l < \omega$.

Claim 8.1 $T_k^{(m)} \vdash d(\Omega^{2+k} \cdot m) \in W_n$.

Proof of the Claim 8.1. By induction on $\ell\alpha$, we show

$$\alpha <_n d(\Omega^{2+k} \cdot m) \rightarrow \alpha \in W_n$$

Cosider the case $\alpha = d\beta <_n d(\Omega^{2+k} \cdot m)$. Then $\beta = \Omega^{2+k} \cdot m' + \sum_{i=0}^{1+k} \Omega^i \cdot \beta_i$ for some $m' < m$, $\beta_i < \Omega$. By the Lemma 8.7 it suffices to show $\{\beta_0, \ldots, \beta_{1+k}\} \subset W_n$. This follows from $\beta_i <_n d(\Omega^{2+k} \cdot m)$ and IH. □
Now the lemma follows from the Claim 8.1 and the IH on l. □

Now by a metainduction on $\ell\alpha$ we have the

Lemma 8.9 *For each $\alpha < d(\Omega^{2+k} \cdot m + \Omega\omega)$,*

$$T_k^{(m)} \vdash \alpha \in W_n$$

Next we define the *lexicographic path order* over a vocabulary having m function symbols of the arity $2+k$. Let $ar(f)$ denote the *arity* of the function symbol f when the symbol f has a fixed arity.

Definition 8.7 $\mathcal{F}_{kQ}^{(m)}$

1. A set $\mathcal{F}_{kQ}^{(m)}$ of function symbols

$$\mathcal{F}_{kQ}^{(m)} =_{df} \{list\} \cup \{A_p : p < m\} \cup \{f_q : q < Q\}$$

   where *list* is varyadic, $ar(A_p) = 2 + k$ for each $p < m$ and $ar(f_q) = 1$ for each $q < Q$.
   Precedence of these symbols is given by

$$list < A_0 < \cdots < A_{m-1} < f_0 < \cdots < f_{Q-1}$$

2. For a given *countable set Var of variables, Term* denotes the *set of terms over* $\mathcal{F}_{kQ}^{(m)} \cup Var$. Applying the symbol *list* to the empty sequence we produce an *individual constant* $0 =_{df} list()$.
   $\mathcal{G} = \mathcal{G}_{kQ}^{(m)}$ denotes the *set of ground (=closed) terms* in *Term*.

Definition 8.8 $s <_{lpo} t$ for $s, t \in Term$.
For sequences $\bar{t} = (t_0, \ldots, t_{n-1})$, $\bar{s} = (s_0, \ldots, s_{l-1})$ of terms, let $\ll_{lpo}$ denote the *multiset extension of* $<_{lpo}$: $\bar{s} \ll_{lpo} \bar{t}$ iff

$$\exists \bar{s}_0, \ldots, \bar{s}_{n-1}[\bar{s} \approx \bar{s}_0 * \cdots * \bar{s}_{n-1} \,\&\, \forall i < n(\bar{s}_i \leq_{lpo} t_i) \,\&\, \exists i < n(\bar{s}_i <_{lpo} t_i)],$$

where $\approx$ denotes the *permutative congruence*, $*$ *concatenation* and

$$(s_0, \ldots, s_{l-1}) <_{lpo} t \Leftrightarrow_{df} \forall j < l(s_j <_{lpo} t).$$

Put $t \equiv g\bar{t}$, $\bar{t} = (t_0, \ldots, t_{n-1})$.
$s <_{lpo} t$ if one of the following conditions is fulfilled:

1. $s \leq_{lpo} t_i$ for some t_i.
2. $s \equiv h\bar{s}$, $\bar{s} = (s_0, \ldots, s_{l-1})$ with $h < g$: $s_j <_{lpo} t$ for each s_j.
3. $s \equiv g\bar{s}$:
   (a) $g = list$:$\bar{s} \ll_{lpo} \bar{t}$
   (b) $g = A_p\,(p < m)$:

$$\exists j < l = n = 2 + k[\forall i < j(s_i = t_i) \,\&\, s_j <_{lpo} t_j \,\&\, \forall i(j < i < l \rightarrow s_i <_{lpo} t]$$

   (c) $g = f_q$, $(q < Q)$: $s_0 <_{lpo} t_0$.

Definition 8.9 The *norm* $| t |$ of a term t.

1. $| v |= 0\,(v \in Var)$
2. $| list(t_1, \ldots, t_n) |= \max(\{n\} \cup \{1+ | t_i |: 1 \leq i \leq n\})$
3. $| A_p(t_{1+k}, \ldots, t_0) |= \max(\{1 + k, p\} \cup \{| t_i |: i < 2 + k\}) + 1$
4. $| f_q(t) |= \max\{1 + k, m, q, | t |\} + 1$

Definition 8.10 *(cf. [6])* $\pi t \in Od$ for a ground term $t \in \mathcal{G}$

1. $\pi list(t_1, \ldots, t_n) = \omega^{\pi t_1} \# \cdots \# \omega^{\pi t_n}$
2. $\pi A_p(t_{1+k}, \ldots, t_0) = d(\Omega^{2+k} \cdot p + \sum_{i=0}^{1+k} \Omega^i \cdot \pi t_i)$
3. $\pi f_q(t) = d(\Omega^{2+k} \cdot m + \Omega \cdot q + \pi t)$

Definition 8.11 (Buchholz [2]) $s <_k t$
Put $t \equiv g\bar{t}$, $\bar{t} = (t_0, \ldots, t_{n-1})$.
$s <_k t$ if one of the following conditions is fulfilled:

1. $s \leq_k t_i$ for some t_i.
2. $s \equiv h\bar{s}$, $\bar{s} = (s_0, \ldots, s_{l-1})$ with $h < g$:
   $s_j <_k t$ for each s_j and $| s |\leq| t | +k$.
3. $s \equiv g\bar{s}$:
   (a) $g = list$:$\bar{s} \ll_k \bar{t}$ with the multiset extension $\ll_k$ of $<_k$ and $| s |\leq| t | +k$.
   (b) $g = A_p\,(p < m)$:

$$\exists j < l = n = 2 + k[\forall i < j(s_i = t_i) \,\&\, s_j <_k t_j \,\&\, \forall i(j < i < l \rightarrow s_i <_k t]$$

$$\text{and } | s |\leq| t | +k.$$

   (c) $g = f_q$, $(q < Q)$: $s_0 <_k t_0$ and $| s |\leq| t | +k$..

Lemma 8.10 *1.* $s <_{lpo} t \rightarrow| s\sigma |\leq| t\sigma | + | s |$ *for any substitution* σ.

2. $s <_{lpo} t \rightarrow s\sigma <_{|s|} t\sigma$ *for any substitution* σ.

3. *If a finite rewrite rule* $\mathcal{R} = \{(l,r)\}$ *over* $\mathcal{F}_{kQ}^{(m)}$ *is reducing under* $<_{lpo}$*, then* $\rightarrow_{\mathcal{R}} \subseteq <_n$ *with* $n = \max\{\mid r \mid : (l,r) \in \mathcal{R}\}$.

4. $\mid t \mid = N\pi t$ *for any ground term* $t \in \mathcal{G}$.

5. $s <_k t \rightarrow \pi s <_k \pi t$ *for* $s, t \in \mathcal{G}$.

6. $\mid t \mid \leq l \rightarrow \pi t <_l d(\Omega^{2+k} \cdot m + \Omega \cdot Q)$ *for* $t \in \mathcal{G}$.

Let $\mathcal{R} = \{(l,r)\}$ be a finite rewrite rule over $\mathcal{F}_{kQ}^{(m)}$ such that $\mathcal{R}$ is reducing under $<_{lpo}$. The *derivation length function* $Dh_{\mathcal{R}}$ is defined by

$$dh_{\mathcal{R}}(t) =_{df} \max\{l \in \omega : \exists(t_0, \ldots, t_l)[t \equiv t_l \rightarrow_{\mathcal{R}} \cdots \rightarrow_{\mathcal{R}} t_0]\}$$
$$Dh_{\mathcal{R}}(n) =_{df} \max\{dh_{\mathcal{R}}(t) : \mid t \mid \leq n\}$$

Lemma 8.11 *The derivation length function* $Dh_{\mathcal{R}}(n)$ *is majorized by the function* $G_n(d(\Omega^{2+k} \cdot m + \Omega \cdot Q))$*, i.e.,*

$$\exists n_0 \forall n \geq n_0 [Dh_{\mathcal{R}}(n) \leq G_n(d(\Omega^{2+k} \cdot m + \Omega \cdot Q))]$$

Proof. By Lemma 8.10 pick an n_0 depending on $\mathcal{R}$ so that $dh_{\mathcal{R}}(t) \leq G_{n_0}(\pi t)$. If $n \geq n_0$ and $\mid t \mid \leq n$, then by Lemma 8.10 again, $\pi t <_n d(\Omega^{2+k} \cdot m + \Omega \cdot Q)$. Thus $dh_{\mathcal{R}}(t) < G_n(d(\Omega^{2+k} \cdot m + \Omega \cdot Q))$ by $<_{n_0} \subseteq <_n$. □

Next we show that the computation of a multiply recursive function $F_{\omega^{1+k} \cdot m}$ $(k, m \in \omega)$ can be regarded as a derivation in a finite rewrite rule. We learnt this view from Hofbauer [13].

For a term t let St denote the term $list(0, t_1, \ldots, t_n)$ if $t \equiv list(t_1, \ldots, t_n)$ and $list(t)$ otherwise. $0^{(m)} = S \cdots S0 = list(0, \ldots, 0)$ is the *mth numeral*. Observe that $\mid 0^{(m)} \mid = \pi 0^{(m)} = m$.

Consider the following interpretation:

$$0 := list(); \ +1 := S; \ F_\alpha(x_0) := A_p(x_{1+k}, \ldots, x_1, x_0)$$

with $\alpha = \omega^{1+k} \cdot p + \sum_{i=0}^{k} \omega^i \cdot x_{i+1}$ $0 \leq p < m$ and

$$F_{\omega^{1+k} \cdot m + (1+q)} := f_q \ (q < Q)$$

Definition 8.12 *Grzegorczyk-Ackermann Rewrite Rule* $\mathcal{R}_Q$ *for* F_α*,* $\alpha \leq \omega^{1+k} \cdot m + Q$

1. $F_\alpha(0) = 2$
   (a) $A_p(\bar{x}, 0) \rightarrow 2 = SS0$
   (b) $f_q(0) \rightarrow 2$

2. $F_{\omega^{1+k} \cdot p + \alpha + x_1 + 1}(x_0 + 1) = A_p(\bar{x}, Sx_1, Sx_0) \rightarrow$
   $A_p(\bar{x}, x_1, A_p(\bar{x}, Sx_1, x_0)) = F_{\omega^{1+k} \cdot p + \alpha + x_1}(F_{\omega^{1+k} \cdot p + \alpha + x_1 + 1}(x_0))$
   with $\alpha = \sum_{i=1}^{k} \omega^i \cdot x_{i+1}$.

3. $f_q(Sx) \rightarrow f_{q-1}(f_q(x))$

4. $F_{\omega^{1+k} \cdot m + 1}(x_0 + 1) = f_0(Sx_0) \rightarrow A_{m-1}(Sf_0(x_0), \bar{0}, f_0(x_0))$
   $= F_{\omega^{1+k} \cdot m}(F_{\omega^{1+k} \cdot m + 1}(x_0))$

5. $F_{\omega^{1+k} \cdot p}(x_0 + 1) = A_p(\bar{0}, Sx_0) \rightarrow A_{p-1}(SSx_0, \bar{0}, Sx_0)$
   $= F_{\omega^{1+k} \cdot (p-1) + \omega^k \cdot (x_0 + 2)}(x_0 + 1) \ (p \neq 0)$

6. $F_{\omega^{1+k} \cdot p + \alpha + \omega^i \cdot (x_{i+1} + 1)}(x_0 + 1) = A_p(\bar{x}, Sx_{i+1}, \bar{0}, Sx_0) \rightarrow$
   $A_p(\bar{x}, x_{i+1}, SSx_0, \bar{0}, Sx_0) = F_{\omega^{1+k} \cdot p + \alpha + \omega^i \cdot x_{i+1} + \omega^{i-1} \cdot (x_0 + 2)}(x_0 + 1)$
   $(i \neq 0)$
   with $\alpha = \sum_{j=i+2}^{1+k} \omega^{j-1} \cdot x_j$

7. $F_0(x_0 + 1) = A_0(\bar{0}, Sx_0) \rightarrow SSA_0(\bar{0}, x_0) = F_0(x_0) + 2$

Definition 8.13 1. $\mathcal{NG}$ denotes the set of ground terms over $0, S, A_p, f_q$.

2. For each $t \in \mathcal{NG}$, $no(t) \in \omega$ is defined by
   (a) $no(0) = 0$

(b) $no(St) = no(t) + 1$

(c) $no(A_p(t_{1+k}, \ldots, t_1, t_0)) = no(f_q(t_0)) = no(t_0)$

Lemma 8.12 *1. The Grzegorczyk-Ackermann rewrite rule $\mathcal{R}_Q$ is reducing under $<_{lpo}$.*

2. For each $(l.r) \in \mathcal{R}_Q$ and each substitution σ with $l\sigma, r\sigma \in \mathcal{NG}$,

$$no(r\sigma) \leq no(l\sigma) + 2$$

3. $\mathcal{R}_Q$ is terminating. Let $\tilde{t}$ denote the unique normal form of $t \in \mathcal{NG}$. Then $\tilde{t}$ is a numeral and $val(t) =_{df} no(\tilde{t})$ denotes the value of the ground term t.

4. For $t \in \mathcal{NG}$,

$$val(t) \leq no(t) + 2dh_{\mathcal{R}_Q}(t)$$

Let $Dh(<_{lpo}, \mathcal{F}_{kq}^{(m)})$ denote the set of derivation lengths functions $Dh_{\mathcal{R}}$ such that $\mathcal{R}$ is a finite rewrite rule over $\mathcal{F}_{kq}^{(m)}$ which is reducing under $<_{lpo}$.

Lemma 8.13 *For each $q < \omega$*

1. $F_{\omega^{1+k} \cdot m+q}$ is elementary recursive in $Dh_{\mathcal{R}_q}$.

2. $F_{\omega^{1+k} \cdot m+q}$ is majorized by the function $G_n(d\eta_{kmq})$ with

$$\eta_{kmq} =_{df} \begin{cases} \Omega^{2+k} \cdot (m-1) + \Omega^{1+k} \cdot \omega & q = 0 \\ \Omega^{2+k} \cdot m + \Omega \cdot q + \omega & \textit{otherwise} \end{cases}$$

Proof. **Case1** $q = 0$:We have, by the Lemma 8.12

$$\begin{aligned} F_{\omega^{1+k} \cdot m}(n) &= F_{\omega^{1+k} \cdot (m-1) + \omega^k \cdot (n+1)}(n) \\ &= val(A_{m-1}(0^{(n+1)}, \bar{0}, 0^{(n)})) \leq n + 2dh_{\mathcal{R}_0}(A_{m-1}(0^{(n+1)}, \bar{0}, 0^{(n)})) \end{aligned}$$

1. For some constant c depending on m, k, $\mid A_{m-1}(0^{(n+1)}, \bar{0}, 0^{(n)}) \mid \leq n + c$. Thus $F_{\omega^{1+k} \cdot m}(n) \leq n + 2Dh_{\mathcal{R}_0}(n + c)$.

2. By the Lemma 8.10 there exists an n_0 such that for any $n \geq n_0$,

$$dh_{\mathcal{R}_0}(A_{m-1}(0^{(n+1)}, \bar{0}, 0^{(n)})) \leq G_n \alpha_n$$

with

$$\alpha_n = \pi(A_{m-1}(0^{(n+1)}, \bar{0}, 0^{(n)})) = d(\Omega^{2+k} \cdot (m-1) + \Omega^{1+k} \cdot (n+1) + n).$$

We show the following Claim which yields $\forall n \geq n_0[F_{\omega^{1+k} \cdot m}(n) \leq G_n d\eta_{km0}]$:

Claim 8.2 $n + 2G_n\alpha_n < G_n d\eta_{km0}$

Proof of the Claim 8.2. We have, by $n + 1 <_n \omega$ and $Nd\eta_{km0} \geq 2$
$n \# \alpha_n \cdot 2 <_n d(\Omega^{2+k} \cdot (m-1) + \Omega^{1+k} \cdot \omega) = d\eta_{km0}$. Also, in general, we have $G_n\alpha + G_n\beta \leq G_n(\alpha \# \beta)$.
From these we see the Claim. □

Case2 $q \neq 0$: We have

$$F_{\omega^{1+k} \cdot m+(1+q)}(n) = val(f_q(0^{(n)})) \leq n + 2dh_{\mathcal{R}_{1+q}}(f_q(0^{(n)})).$$

1. For a constant c depending on m, k, q, $\mid f_q(0^{(n)}) \mid \leq n + c$.

2. As in the **Case 1**, there exists an n_0 such that for any $n \geq n_0$,

$$dh_{\mathcal{R}_{1+q}}(f_q(0^{(n)})) \leq G_n\alpha_n \text{ with } \alpha_n = \pi f_q(0^{(n)}) = d(\Omega^{2+k} \cdot m + \Omega \cdot q + n).$$

We have $n \# \alpha_n \cdot 2 <_n d(\Omega^{2+k} \cdot m + \Omega \cdot q + \omega) = d\eta_{kmq}$. Thus for any $n \geq n_0$ $F_{\omega^{1+k} \cdot m+(1+q)}(n) < G_n d\eta_{kmq}$.

□

Theorem 8.2 *For each k,m with $0 \leq k < \omega$, $0 < m < \omega$,*

$$T_k^{(m)} = I\Sigma_1 + TIR^{(m)}[\omega^{1+k}, \Pi_2] = I\Sigma_1 + C_1^{I\Sigma_1}(\underline{\omega^k \cdot m}) = I\Sigma_1 + F_{\omega^{1+k}\cdot m} \downarrow$$

and

$$\begin{aligned}
&Tot(T_k^{(m)}) = \\
&PR(F_{\omega^{1+k}\cdot m}) = ER(\{F_{\omega^{1+k}\cdot m+q} : q < \omega\}) = \\
&PR(Dh(<_{lpo}, \mathcal{F}_{k0}^{(m)})) = ER(\{Dh(<_{lpo}, \mathcal{F}_{kq}^{(m)}) : q < \omega\}) = \\
&PR(G_n(d(\Omega^{2+k}\cdot(m-1) + \Omega^{1+k}\cdot\omega))) = ER(\{G_n(d(\Omega^{2+k}\cdot m + \Omega\cdot q)) : q < \omega\})
\end{aligned}$$

Also these classes of functions are majorized by any one of the functions $F_{\omega^{1+k}\cdot m+\omega}$ and $G_n(d(\Omega^{2+k}\cdot m+\Omega\cdot\omega))$.

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